GOVERNMENT OF INDIA
DEPARTMENT OF ARCHAEOLOGY
CENTRAL ARCHAEOLOGICAL LIBRARY

CLASS

CALL No 510.949 Tho
Vol. 1

D.G.A. 79.

METROPOLITAN BOOK CO. LTD.
1, FAIZ BAZAR
DELHI

THE LOEB CLASSICAL LIBRARY
FOUNDED BY JAMES LOEB, LL.D.

EDITED BY
† T. E. PAGE, C.H., LITT.D.
† E. CAPPS, PH.D., LL.D. † W. H. D. ROUSE, LITT.D.
L. A. POST, M.A. E. H. WARMINGTON, M.A., F.R.HIST.SOC.

GREEK MATHEMATICS

I

SELECTIONS

ILLUSTRATING THE HISTORY OF

GREEK MATHEMATICS

WITH AN ENGLISH TRANSLATION BY

IVOR THOMAS

FORMERLY SCHOLAR OF ST. JOHN'S AND SENIOR DEMY
OF MAGDALEN COLLEGE, OXFORD

915

IN TWO VOLUMES

I

FROM THALES TO EUCLID

510.938
Tho

LONDON
WILLIAM HEINEMANN LTD
CAMBRIDGE, MASSACHUSETTS
HARVARD UNIVERSITY PRESS
MCMLI

First printed 1939
Revised and reprinted 1951

CENTRAL ARCHAEOLOGICAL
LIBRARY, NEW DELHI.
Acc. No. 915
Date. 28-1-54
Call No. 510.949 / ~~Gre~~ / Tho.

Printed in Great Britain

TO

M. E. B.

PREFACE

THE story of Greek mathematics is the tale of one of the most stupendous achievements in the history of human thought. It is my hope that these selections, which furnish a reasonably complete picture of the rise of Greek mathematics from earliest days, will be found useful alike by classical scholars, desiring easy access to a most characteristic aspect of the Greek genius, and by mathematicians, anxious to learn something about the origins of their science. In these days of specialization the excellent custom which formerly prevailed at Oxford and Cambridge whereby men took honours both in classics and in mathematics has gone by the board. It is now rare to find a classical scholar with even an elementary knowledge of mathematics, and the mathematician's knowledge of Greek is usually confined to the letters of the alphabet. By presenting the main Greek sources side by side with an English translation, reasonably annotated, I trust I have done something to bridge the gap.

For the classical scholar Greek mathematics is a brilliant after-glow which lightened the sky long after the sun of Hellas had set. Greek mathematics sprang from the same impulse as Greek philosophy, but Greek philosophy reached its maturity in the fourth century before Christ, the century of Plato and Aristotle, and thereafter never spoke with like con-

viction until the voice of Plato became reincarnate in the schools of Egypt. Yet such was the vitality of Hellenic thought that the autumn flowering of Greek philosophy in Aristotle was only the spring of Greek mathematics. It was Euclid, following hard on the heels of Aristotle in point of time, but teaching in distant Alexandria, who first transformed mathematics from a number of uncoordinated and loosely-proved theorems into an articulated and surely-grounded science; and in the succeeding hundred years Archimedes and Apollonius raised mathematics to heights not surpassed till the sixteenth century of the Christian era.

To the mathematician his Greek predecessors are deserving of study in that they laid the foundations on which all subsequent mathematical science is based. Names still in everyday use testify to this origin—Euclidean geometry, Pythagoras's theorem, Archimedes' axiom, the quadratrix of Hippias or Dinostratus, the cissoid of Diocles, the conchoid of Nicomedes. I cannot help feeling that mathematicians will welcome the opportunity of learning the reasons for these names, and that the extracts which follow will enable them to do so more easily than is now possible. In perusing these extracts they will doubtless be impressed by three features. The first is the rigour with which the great Greek geometers demonstrated what they set out to prove. This is most noticeable in their treatment of the indefinitely small, a subject whose pitfalls had been pointed out by Zeno in four arguments of remarkable acuteness. Archimedes, for example, carries out operations equivalent to the integral calculus, but he refuses to posit the existence of infinitesimal quanti-

ties, and avoids logical errors which infected the calculus until quite recent times. The second feature of Greek mathematics which will impress the modern student is the dominating position of geometry. Early in the present century there was a powerful movement for the "arithmetization" of all mathematics. Among the Greeks there was a similar impulse towards the "geometrization" of all mathematics. Magnitudes were from earliest times represented by straight lines, and the Pythagoreans developed a geometrical algebra performing operations equivalent to the solution of equations of the second degree. Later Archimedes evaluated by purely geometrical means the area of a variety of surfaces, and Apollonius developed his awe-inspiring geometrical theory of the conic sections. The third feature which cannot fail to impress a modern mathematician is the perfection of form in the work of the great Greek geometers. This perfection of form, which is another expression of the same genius that gave us the Parthenon and the plays of Sophocles, is found equally in the proof of individual propositions and in the ordering of those separate propositions into books; it reaches its height, perhaps, in the *Elements* of Euclid.

In making the selections which follow I have drawn not only on the ancient mathematicians but on many other writers who can throw light on the history of Greek mathematics. Thanks largely to the labours of a band of Continental scholars, admirable standard texts of most Greek mathematical works now exist, and I have followed these texts, indicating only the more important variants and emendations. In the selection of the passages, in their arrangement and at

innumerable points in the translation and notes I owe an irredeemable debt of gratitude to the works of Sir Thomas Heath, who has been good enough, in addition, to answer a number of queries on specific points. These works, covering almost every aspect of Greek mathematics and astronomy, are something of which English scholarship may justly feel proud. His *History of Greek Mathematics* is unexcelled in any language. Yet there may still be room for a work which will give the chief sources in the original Greek together with a translation and sufficient notes.

In a strictly logical arrangement the passages would, no doubt, be grouped wholly by subjects or by persons. But such an arrangement would not be satisfactory. I imagine that the average reader would like to see, for example, all the passages on the squaring of the circle together, but would also like to see the varied discoveries of Archimedes in a single section. The arrangement here adopted is a compromise for which I must ask the reader's indulgence where he might himself have made a different grouping. The contributions of the Greeks to arithmetic, geometry, trigonometry, mensuration and algebra are noticed as fully as possible, but astronomy and music, though included by the Greeks under the name mathematics, have had to be almost wholly excluded.

I am greatly indebted to Messrs. R. and R. Clark for the skill and care shown in the difficult task of making this book.

I. T.

Adelphi, London
April 1939

CONTENTS OF VOLUME I

CONTENTS

ABBREVIATIONS

Heath, *H.G.M.*	Sir Thomas Heath, *A History of Greek Mathematics*, 2 vols., Oxford 1921.
Diels, *Vors.*[5]	Hermann Diels, *Die Fragmente der Vorsokratiker*, 3 vols., 5th ed., edited by Walther Kranz, Berlin 1934–1937.

Both cited by volume and page.

References to modern editions of classical texts are by volume (where necessary), page and line, *e.g.*, Eucl. ed. Heiberg-Menge vii. 14. 1—16. 5 refers to *Euclidis Opera Omnia*, edited by I. L. Heiberg and H. Menge, vol. vii., page 14 line 1 to page 16 line 5.

I. INTRODUCTORY

I. INTRODUCTORY

(a) Mathematics and its Divisions

(i.) *Origin of the Name*

Anatolius ap. Her. *Def.*, ed. Heiberg 160. 8–162. 2

Ἐκ τῶν Ἀνατολίου . . .

" Ἀπὸ τίνος δὲ μαθηματικὴ ὠνομάσθη;

" Οἱ μὲν ἀπὸ τοῦ Περιπάτου φάσκοντες ῥητορικῆς μὲν καὶ ποιητικῆς συμπάσης τε τῆς δημώδους μουσικῆς δύνασθαί τινα συνεῖναι καὶ μὴ μαθόντα, τὰ δὲ καλούμενα ἰδίως μαθήματα οὐδένα εἰς εἴδησιν λαμβάνειν μὴ οὐχὶ πρότερον ἐν μαθήσει γενόμενον τούτων, διὰ τοῦτο μαθηματικὴν καλεῖσθαι τὴν περὶ τούτων θεωρίαν ὑπελάμβανον. θέσθαι δὲ λέγονται τὸ τῆς μαθηματικῆς ὄνομα ἰδιαίτερον ἐπὶ μόνης γεωμετρίας καὶ ἀριθμητικῆς οἱ ἀπὸ τοῦ Πυθαγόρου· τὸ γὰρ πάλαι χωρὶς ἑκατέρα τούτων ὠνομάζετο, κοινὸν δὲ οὐδὲν ἦν ἀμφοῖν ὄνομα."

[a] Anatolius was bishop of Laodicea about A.D. 280. In a letter by Michael Psellus he is said to have written a concise treatise on the Egyptian method of reckoning.

[b] *i.e.* singing or playing, as opposed to the mathematical study of musical intervals.

[c] The word μάθημα, from μαθεῖν, means in the first place " that which is learnt." In Plato it is used in the general sense for any subject of study or instruction, but with a tendency to restrict it to the studies now called mathematics. By the time of Aristotle this restriction had become established.

I. INTRODUCTORY

(a) Mathematics and its Divisions

(i.) *Origin of the Name*

Anatolius, cited by Heron, *Definitions*, ed. Heiberg 160. 8–162. 2

From the works of Anatolius [a] . . .

" Why is mathematics so named ?

" The Peripatetics say that rhetoric and poetry and the whole of popular music [b] can be understood without any course of instruction, but no one can acquire knowledge of the subjects called by the special name *mathematics* unless he has first gone through a course of instruction in them ; and for this reason the study of these subjects was called *mathematics*.[c] The Pythagoreans are said to have given the special name *mathematics* only to geometry and arithmetic ; previously each had been called by its separate name, and there was no name common to both." [d]

[a] The esoteric members of the Pythagorean school, who had learnt the Pythagorean theory of knowledge in its entirety, are said to have been called *mathematicians* (μαθηματικοί), whereas the exoteric members, who merely knew the Pythagorean rules of conduct, were called *hearers* (ἀκουσματικοί). See Iamblichus, *De Vita Pythag.* 18. 81, ed. Deubner 46. 24 ff.

(ii.) *The Pythagorean Quadrivium*

Archytas ap. Porphyr. *in Ptol. Harm.*, ed. Wallis, *Opera Math.* iii. 236. 40–237. 1; Diels, *Vors.* i[5]. 431. 26–432. 8

Παρακείσθω δὲ καὶ νῦν τὰ Ἀρχύτα τοῦ Πυθαγορείου, οὗ μάλιστα καὶ γνήσια λέγεται εἶναι τὰ συγγράμματα· λέγει δὲ ἐν τῷ Περὶ μαθηματικῆς εὐθὺς ἐναρχόμενος τοῦ λόγου τάδε.

"Καλῶς μοι δοκοῦντι τοὶ περὶ τὰ μαθήματα διαγνώμεναι, καὶ οὐδὲν ἄτοπον ὀρθῶς αὐτούς, οἷά ἐντι, περὶ ἑκάστων φρονέειν· περὶ γὰρ τᾶς τῶν ὅλων φύσιος καλῶς διαγνόντες ἔμελλον καὶ περὶ τῶν κατὰ μέρος, οἷά ἐντι, καλῶς ὀψεῖσθαι. περί τε δὴ τᾶς τῶν ἄστρων ταχυτᾶτος καὶ ἐπιτολᾶν καὶ δυσίων παρέδωκαν ἁμῖν σαφῆ διάγνωσιν καὶ περὶ γαμετρίας καὶ ἀριθμῶν καὶ σφαιρικᾶς καὶ οὐχ ἥκιστα περὶ μωσικᾶς. ταῦτα γὰρ τὰ μαθήματα δοκοῦντι ἦμεν ἀδελφεά."

---

[a] Archytas lived in the first half of the fourth century B.C. at Taras (Tarentum) in Magna Graecia. He is said to have dissuaded Dionysius from putting Plato to death. For seven years he commanded the forces of his city-state, though the law forbade anyone to hold the post normally for more than one year, and he was never defeated. He is said to have been the first to write on mechanics, and to have invented a mechanical dove which would fly. For such of his mathematical discoveries as have survived, see pp. 112-115, 130-133, 284-289.

(ii.) *The Pythagorean Quadrivium*

Archytas, cited by Porphyry in his *Commentary on Ptolemy's Harmonics*, ed. Wallis, *Opera Mathematica* iii. 236. 40–237. 1; Diels, *Vors.* i[5]. 431. 26–432. 8

Let us now cite the words of Archytas[a] the Pythagorean, whose writings are said to be mainly authentic. In his book *On Mathematics* right at the beginning of the argument he writes thus:

"The mathematicians seem to me to have arrived at true knowledge, and it is not surprising that they rightly conceive the nature of each individual thing; for, having reached true knowledge about the nature of the universe as a whole, they were bound to see in its true light the nature of the parts as well. Thus they have handed down to us clear knowledge about the speed of the stars, and their risings and settings, and about geometry, arithmetic and sphaeric, and, not least, about music; for these studies appear to be sisters."[b]

[b] *Sphaeric* is clearly identical with astronomy, and is aptly defined by Heath, *H.G.M.* i. 11 as "the geometry of the sphere considered solely with reference to the problem of accounting for the motions of the heavenly bodies." The same *quadrivium* is attributed to the Pythagoreans by Nicomachus, Theon of Smyrna and Proclus, but in the order arithmetic, music, geometry and sphaeric. The logic of this order is that arithmetic and music are concerned with number (ποσόν), arithmetic with number in itself and music with number in relation to sounds; while geometry and sphaeric are concerned with magnitude (πηλίκον), geometry with magnitude at rest, sphaeric with magnitude in motion.

(iii.) *Plato's Scheme*

Plat. *Rep.* vii. 525 A–530 D

(*a*) *Logistic and Arithmetic*

'Αλλὰ μὴν λογιστική τε καὶ ἀριθμητικὴ περὶ ἀριθμὸν πᾶσα.

Καὶ μάλα.

Ταῦτα δέ γε φαίνεται ἀγωγὰ πρὸς ἀλήθειαν.

Ὑπερφυῶς μὲν οὖν.

Ὧν ζητοῦμεν ἄρα, ὡς ἔοικε, μαθημάτων ἂν εἴη· πολεμικῷ μὲν γὰρ διὰ τὰς τάξεις ἀναγκαῖον μαθεῖν ταῦτα, φιλοσόφῳ δὲ διὰ τὸ τῆς οὐσίας ἁπτέον εἶναι γενέσεως ἐξαναδύντι, ἢ μηδέποτε λογιστικῷ γενέσθαι. . . .

Τί οὖν οἴει, ὦ Γλαύκων, εἴ τις ἔροιτο αὐτούς· "Ὦ θαυμάσιοι, περὶ ποίων ἀριθμῶν διαλέγεσθε, ἐν οἷς τὸ ἓν οἷον ὑμεῖς ἀξιοῦτέ ἐστιν, ἴσον τε ἕκαστον πᾶν παντὶ καὶ οὐδὲ σμικρὸν διαφέρον, μόριόν τε ἔχον ἐν ἑαυτῷ οὐδέν;" τί ἂν οἴει αὐτοὺς ἀποκρίνασθαι;

Τοῦτο ἔγωγε, ὅτι περὶ τούτων λέγουσιν ὧν διανοηθῆναι μόνον ἐγχωρεῖ, ἄλλως δ' οὐδαμῶς μεταχειρίζεσθαι δυνατόν. . . .

Τί δέ; τόδε ἤδη ἐπεσκέψω, ὡς οἵ τε φύσει

---

[a] The passage is taken from the section dealing with the education of the Guardians. The speakers in the dialogue are Socrates and Glaucon. It is made clear in *Rep.* 537 B-D that the Guardians would receive their chief mathematical training between the ages of twenty and thirty, after two or three years spent in the study of music and gymnastic and as a preliminary to five years' study of dialectic. Plato's scheme, it will be noticed, is virtually identical with the Pythagorean *quadrivium* except for the addition of stereo-

(iii.) *Plato's Scheme*

Plato, *Republic* vii. 525 A–530 D [a]

(a) *Logistic and Arithmetic*

Now logistic and arithmetic treat of the whole of number.

Yes.

And, apparently, they lead us towards truth.

They do, indeed.

It would appear, therefore, that they must be among the studies we seek; for the soldier finds it necessary to learn them in order to draw up his troops, and the philosopher because he is bound to rise out of Becoming and cling to Being on pain of never becoming a reasoner. . . .[b]

Now what would you expect, Glaucon, if someone were to ask them: " My good people, what kind of numbers are you discussing? What are these numbers such as you describe, every unit being equal, each to each, without the smallest difference, and containing within itself no part?" What answer would you expect them to make?

I should expect them to say that the numbers they discuss are capable of being conceived only in thought, and can be dealt with in no other way. . . .

Again; have you ever noticed that those who are

metry; and the addition is more formal than real since stereometrical problems were certainly investigated by the Pythagoreans—not least by Archytas—as part of geometry. Plato also distinguishes *logistic* from *arithmetic* (for which see the extract given below on pp. 16-19), and speaks of *harmonics* (ἁρμονία) not *music* (μουσική), thus avoiding confusion with *popular music* (τὸ δημῶδες μουσικόν).

[b] There is a play on the Greek word, which could mean either " reasoner " or " calculator."

λογιστικοὶ εἰς πάντα τὰ μαθήματα ὡς ἔπος εἰπεῖν ὀξεῖς φύονται, οἵ τε βραδεῖς, ἂν ἐν τούτῳ παιδευθῶσιν καὶ γυμνάσωνται, κἂν μηδὲν ἄλλο ὠφεληθῶσιν, ὅμως εἴς γε τὸ ὀξύτεροι αὐτοὶ αὑτῶν γίγνεσθαι πάντες ἐπιδιδόασιν;

Ἔστιν, ἔφη, οὕτω.

Καὶ μήν, ὡς ἐγᾦμαι, ἅ γε μείζω πόνον παρέχει μανθάνοντι καὶ μελετῶντι, οὐκ ἂν ῥᾳδίως οὐδὲ πολλὰ ἂν εὕροις ὡς τοῦτο.

Οὐ γὰρ οὖν.

Πάντων δὴ ἕνεκα τούτων οὐκ ἀφετέον τὸ μάθημα, ἀλλ' οἱ ἄριστοι τὰς φύσεις παιδευτέοι ἐν αὐτῷ.

Σύμφημι, ἦ δ' ὅς.

(β) *Geometry*

Τοῦτο μὲν τοίνυν, εἶπον, ἓν ἡμῖν κείσθω· δεύτερον δὲ τὸ ἐχόμενον τούτου σκεψώμεθα ἆρά τι προσήκει ἡμῖν.

Τὸ ποῖον; ἢ γεωμετρίαν, ἔφη, λέγεις;

Αὐτὸ τοῦτο, ἦν δ' ἐγώ.

Ὅσον μέν, ἔφη, πρὸς τὰ πολεμικὰ αὐτοῦ τείνει, δῆλον ὅτι προσήκει. . . .

Ἀλλ' οὖν δή, εἶπον, πρὸς μὲν τὰ τοιαῦτα καὶ βραχύ τι ἂν ἐξαρκοῖ γεωμετρίας τε καὶ λογισμῶν μόριον· τὸ δὲ πολὺ αὐτῆς καὶ πορρωτέρω προϊὸν σκοπεῖσθαι δεῖ εἴ τι πρὸς ἐκεῖνο τείνει, πρὸς τὸ ποιεῖν κατιδεῖν ῥᾷον τὴν τοῦ ἀγαθοῦ ἰδέαν. . . . οὐ τοίνυν τοῦτό γε, ἦν δ' ἐγώ, ἀμφισβητήσουσιν ἡμῖν ὅσοι καὶ σμικρὰ γεωμετρίας ἔμπειροι, ὅτι

by nature apt at calculation are—not to make a short matter long—naturally sharp at all studies, and that the slower-witted, if they be trained and exercised in this discipline, even supposing they derive no other advantage from it, at any rate all progress so far as to become sharper than they were before?

Yes, that is true, he said.

And I am of opinion, also, that you would not easily find many sciences which give the learner and the student greater trouble than this.

No, indeed.

For all these reasons, then, this study must not be rejected, but all the finest spirits must be educated in it.[a]

I agree, he said.

(β) Geometry

Then let us consider this, I said, as one point settled. In the second place let us examine whether the science bordering on arithmetic concerns us.

What is that? Do you mean geometry? he said.

Exactly, I replied.

So far as it bears on military matters, he said, it obviously concerns us. . . .

But for these purposes, I observed, a trifling knowledge of geometry and calculations would suffice; what we have to consider is whether a more thorough and advanced study of the subject tends to facilitate contemplation of the Idea of the Good. . . . Well, even those who are only slightly conversant with geometry will not dispute us in saying that this

[a] Plato's final reason may strike contemporary educationists as somewhat odd.

αὕτη ἡ ἐπιστήμη πᾶν τοὐναντίον ἔχει τοῖς ἐν αὐτῇ λόγοις λεγομένοις ὑπὸ τῶν μεταχειριζομένων.

Πῶς; ἔφη.

Λέγουσι μέν που μάλα γελοίως τε καὶ ἀναγκαίως· ὡς γὰρ πράττοντές τε καὶ πράξεως ἕνεκα πάντας τοὺς λόγους ποιούμενοι λέγουσιν τετραγωνίζειν τε καὶ παρατείνειν καὶ προστιθέναι καὶ πάντα οὕτω φθεγγόμενοι, τὸ δ' ἔστι που πᾶν τὸ μάθημα γνώσεως ἕνεκα ἐπιτηδευόμενον. . . .

(γ) *Stereometry*

Τί δέ; τρίτον θῶμεν ἀστρονομίαν; ἢ οὐ δοκεῖ;

Ἐμοὶ γοῦν, ἔφη. . . .

Νυνδὴ γὰρ οὐκ ὀρθῶς τὸ ἑξῆς ἐλάβομεν τῇ γεωμετρίᾳ.

Πῶς λαβόντες; ἔφη.

Μετὰ ἐπίπεδον, ἦν δ' ἐγώ, ἐν περιφορᾷ ὂν ἤδη στερεὸν λαβόντες, πρὶν αὐτὸ καθ' αὐτὸ λαβεῖν· ὀρθῶς δὲ ἔχει ἑξῆς μετὰ δευτέραν αὔξην τρίτην λαμβάνειν. ἔστι δέ που τοῦτο περὶ τὴν τῶν κύβων αὔξην καὶ τὸ βάθους μετέχον.

Ἔστι γάρ, ἔφη· ἀλλὰ ταῦτά γε, ὦ Σώκρατες, δοκεῖ οὔπω ηὑρῆσθαι.

Διττὰ γάρ, ἦν δ' ἐγώ, τὰ αἴτια· ὅτι τε οὐδεμία πόλις ἐντίμως αὐτὰ ἔχει, ἀσθενῶς ζητεῖται χαλεπὰ ὄντα, ἐπιστάτου τε δέονται οἱ ζητοῦντες, ἄνευ οὗ οὐκ ἂν εὕροιεν, ὃν πρῶτον μὲν γενέσθαι χαλεπόν,

---

[a] It is useful to know that these terms, which are regularly found in Euclid, were already in technical use in Plato's day.

[b] Lit. "increase of cubes," where the word "increase" is the same as that translated above by "dimension."

science holds a position the very opposite from that implied in the language of those who practise it.

How so ? he asked.

They speak, I gather, in an exceedingly ridiculous and poverty-stricken way. For they fashion all their arguments as though they were engaged in business and had some practical end in view, speaking of squaring and producing and adding[a] and so on, whereas in reality, I fancy, the study is pursued wholly for the sake of knowledge. . . .

(γ) *Stereometry*

Again ; shall we put astronomy third, or do you think otherwise ?

That suits me, he said. . . .

We were wrong just now in what we took as the study next in order after geometry.

What did we take ? he asked.

After dealing with plane surfaces, I replied, we proceeded to consider solids in motion before considering solids in themselves ; the correct procedure, after the second dimension, is to consider the third dimension. This brings us, I believe, to cubical increase[b] and to figures partaking of depth.

Yes, he replied ; but these subjects, Socrates, do not appear to have been yet investigated.

The reasons, I said, are twofold. In the first place, no state holds them in honour and so, being difficult, they are investigated only in desultory manner. In the second place, the investigators lack a director, and without such a person they will make no discoveries. Now to find such a person is a diffi-

There is probably a playful reference to the problem of doubling the cube, for which see *infra*, pp. 256-309.

ἔπειτα καὶ γενομένου, ὡς νῦν ἔχει, οὐκ ἂν πείθοντο οἱ περὶ ταῦτα ζητητικοὶ μεγαλοφρονούμενοι. εἰ δὲ πόλις ὅλη συνεπιστατοῖ ἐντίμως ἄγουσα αὐτά, οὗτοί τε ἂν πείθοιντο καὶ συνεχῶς τε ἂν καὶ ἐντόνως ζητούμενα ἐκφανῆ γένοιτο ὅπῃ ἔχει· ἐπεὶ καὶ νῦν ὑπὸ τῶν πολλῶν ἀτιμαζόμενα καὶ κολουόμενα, ὑπὸ δὲ τῶν ζητούντων λόγον οὐκ ἐχόντων καθ' ὅτι χρήσιμα, ὅμως πρὸς ἅπαντα ταῦτα βίᾳ ὑπὸ χάριτος αὐξάνεται, καὶ οὐδὲν θαυμαστὸν αὐτὰ φανῆναι.

Καὶ μὲν δή, ἔφη, τό γε ἐπίχαρι καὶ διαφερόντως ἔχει. ἀλλά μοι σαφέστερον εἰπὲ ἃ νυνδὴ ἔλεγες. τὴν μὲν γάρ που τοῦ ἐπιπέδου πραγματείαν γεωμετρίαν ἐτίθεις.

Ναί, ἦν δ' ἐγώ.

Εἶτά γ', ἔφη, τὸ μὲν πρῶτον ἀστρονομίαν μετὰ ταύτην, ὕστερον δ' ἀνεχώρησας.

Σπεύδων γάρ, ἔφην, ταχὺ πάντα διεξελθεῖν μᾶλλον βραδύνω· ἑξῆς γὰρ οὖσαν τὴν βάθους αὔξης μέθοδον, ὅτι τῇ ζητήσει γελοίως ἔχει, ὑπερβὰς αὐτὴν μετὰ γεωμετρίαν ἀστρονομίαν ἔλεγον, φορὰν οὖσαν βάθους.

Ὀρθῶς, ἔφη, λέγεις.

---

[a] These words (ὡς νῦν ἔχει) can be taken either with what goes before or with what comes after. In the former case Plato (or Socrates) will be referring to a distinguished contemporary (such as Eudoxus or Archytas) who had already made discoveries in solid geometry.

[b] This passage has been thought to have some bearing on the question whether the Socrates of the dialogue is meant to be the Socrates of history or not. The condition of stereometry, as described in the dialogue, certainly does not fit

cult task, and even supposing one appeared on the scene, as matters now stand,[a] those who are investigating these problems, being swollen with pride, would pay no heed to him. But if a whole state were to honour this study and constitute itself the director thereof, they would pay heed, and the subject, being continuously and earnestly investigated, would be brought to light. For even now, neglected and curtailed as it is, not only by the many but even by professed students, who can suggest no use for it, nevertheless in the face of all these obstacles it makes progress on account of its elegance, and it would not be astonishing if it were fully unravelled.

It is certainly an exceedingly fascinating subject, he said. But pray tell me more clearly what you were saying just now. I think you defined geometry as the investigation of plane surfaces.

Yes, I said.

Then, he observed, you first placed astronomy after it, but later drew back.

The more I hasten to cover the ground, I said, the more slowly I travel; the study of solid bodies comes next in order, but because of the absurd way in which it is investigated I passed it over and spoke of astronomy, which involves the motion of solid bodies, as next after geometry.

You are quite right, he said.[b]

Plato's generation, when Archytas and Eudoxus were making brilliant discoveries in solid geometry; but, even during the lifetime of Socrates, Democritus and Hippocrates had made notable contributions to the same science. This passage cannot help, therefore, towards the solution of that problem. All that Plato meant, it would appear, was that stereometry had not been made a formal element in the curriculum but was treated as part of geometry.

(δ) *Astronomy*

Τέταρτον τοίνυν, ἦν δ᾽ ἐγώ, τιθῶμεν μάθημα ἀστρονομίαν, ὡς ὑπαρχούσης τῆς νῦν παραλειπομένης, ἐὰν αὐτὴν πόλις μετίῃ. . . . ταῦτα μὲν τὰ ἐν τῷ οὐρανῷ ποικίλματα, ἐπείπερ ἐν ὁρατῷ πεποίκιλται, κάλλιστα μὲν ἡγεῖσθαι καὶ ἀκριβέστατα τῶν τοιούτων ἔχειν, τῶν δὲ ἀληθινῶν πολὺ ἐνδεῖν, ἃς τὸ ὂν τάχος καὶ ἡ οὖσα βραδυτὴς ἐν τῷ ἀληθινῷ ἀριθμῷ καὶ πᾶσι τοῖς ἀληθέσι σχήμασι φοράς τε πρὸς ἄλληλα φέρεται καὶ τὰ ἐνόντα φέρει, ἃ δὴ λόγῳ μὲν καὶ διανοίᾳ ληπτά, ὄψει δ᾽ οὔ· ἢ σὺ οἴει;

Οὐδαμῶς γε, ἔφη.

Οὐκοῦν, εἶπον, τῇ περὶ τὸν οὐρανὸν ποικιλίᾳ παραδείγμασι χρηστέον τῆς πρὸς ἐκεῖνα μαθήσεως ἕνεκα, ὁμοίως ὥσπερ ἂν εἴ τις ἐντύχοι ὑπὸ Δαιδάλου ἤ τινος ἄλλου δημιουργοῦ ἢ γραφέως διαφερόντως γεγραμμένοις καὶ ἐκπεπονημένοις διαγράμμασιν. . . . προβλήμασιν ἄρα, ἦν δ᾽ ἐγώ, χρώμενοι ὥσπερ γεωμετρίαν οὕτω καὶ ἀστρονομίαν μέτιμεν, τὰ δ᾽ ἐν τῷ οὐρανῷ ἐάσομεν, εἰ μέλλομεν ὄντως ἀστρονομίας μεταλαμβάνοντες χρήσιμον τὸ φύσει φρόνιμον ἐν τῇ ψυχῇ ἐξ ἀχρήστου ποιήσειν. . . .

---

[a] There seems little doubt that in this passage Plato wished astronomy to be regarded as the pure science of bodies in motion, of which the heavenly bodies could at best afford only one example. Burnet has made desperate efforts to save Plato from himself. According to his contention, Plato meant that astronomy should deal with the true, as opposed to the apparent, motions of the heavenly bodies ; it is tempt-

(δ) *Astronomy*

Let us then put astronomy as the fourth study, regarding that now passed over as waiting only until some state shall take it up. . . . Those broideries yonder in the heaven, forasmuch as they are broidered on a visible ground, are rightly held to be the most beautiful and perfect of visible things, but they are nevertheless far inferior to those that are true, far inferior to those revolutions which absolute speed and absolute slowness, in true number and in all true forms, accomplish relatively to each other, carrying their contents with them—which can indeed be grasped by reason and intelligence, but not by sight. Or do you think otherwise?

No, indeed, he replied.

Therefore, I said, we should use the broideries round the heaven as examples to help the study of those true objects, just as we might use, if we met with them, diagrams surpassingly well drawn and elaborated by Daedalus or any other artist. . . . Hence, I said, we shall approach astronomy, as we do geometry, by means of problems, but we shall leave the starry heavens alone, if we wish to obtain a real grasp of astronomy, and by that means to make useful, instead of useless, the natural intelligence of the soul. . . .[a]

ing but difficult to reconcile this with the decisive language of the text. Fortunately Plato's own pupils in the Academy, notably Eudoxus and Heraclides of Pontus, adopted a different attitude, using mathematics to account for the actual motion of the heavenly bodies; and Plato himself does not appear to have held consistently to the belief here expressed, for he is said to have put to his pupils the question by what combination of uniform circular revolutions the apparent movements of the heavenly bodies can be explained.

(*c*) *Harmonics*

Κινδυνεύει, ἔφην, ὡς πρὸς ἀστρονομίαν ὄμματα πέπηγεν, ὡς πρὸς ἐναρμόνιον φορὰν ὦτα παγῆναι, καὶ αὗται ἀλλήλων ἀδελφαί τινες αἱ ἐπιστῆμαι εἶναι, ὡς οἵ τε Πυθαγόρειοί φασι καὶ ἡμεῖς, ὦ Γλαύκων, συγχωροῦμεν.

(iv.) *Logistic*

Schol. in Plat. *Charm.* 165 E

Λογιστική ἐστι θεωρία τῶν ἀριθμητῶν, οὐχὶ δὲ τῶν ἀριθμῶν μεταχειριστική, οὐ τὸν ὄντως ἀριθμὸν λαμβάνουσα, ἀλλ' ὑποτιθεμένη τὸ μὲν ἓν ὡς μονάδα, τὸ δὲ ἀριθμητὸν ὡς ἀριθμόν, οἷον τὰ τρία τριάδα εἶναι καὶ τὰ δέκα δεκάδα· ἐφ' ὧν ἐπάγει τὰ κατὰ ἀριθμητικὴν θεωρήματα. θεωρεῖ οὖν τοῦτο μὲν τὸ κληθὲν ὑπ' Ἀρχιμήδους βοεικὸν πρόβλημα, τοῦτο δὲ μηλίτας καὶ φιαλίτας ἀριθμούς, τοὺς μὲν ἐπὶ φιαλῶν, τοὺς δὲ ἐπὶ ποίμνης· καὶ ἐπ' ἄλλων δὲ γενῶν τὰ πλήθη τῶν αἰσθητῶν σωμάτων σκοποῦσα, ὡς περὶ τελείων ἀποφαίνεται. ὕλη δὲ αὐτῆς πάντα τὰ ἀριθμητά· μέρη δὲ αὐτῆς αἱ Ἑλληνικαὶ καὶ Αἰγυπτιακαὶ καλούμεναι μέθοδοι ἐν πολλαπλασια-

---

[a] See the fragment from Archytas, *supra*, pp. 4-5.

[b] Socrates proceeds to censure the Pythagoreans for committing the same error as the astronomers: they investigate the numerical ratios subsisting between audible concords, but do not apply themselves to problems, in order to examine what numbers are consonant and what not, and to find out the reason for the difference (ἐπισκοπεῖν τίνες σύμφωνοι ἀριθμοὶ καὶ τίνες οὔ, καὶ διὰ τί ἑκάτεροι).

[c] In the cattle-problem Archimedes sets himself to find the number of bulls and cows of each of four colours. The problem, stripped of its trimmings, is to find eight unknown

(*e*) *Harmonics*

It would appear, I said, that just as our eyes were intended for astronomy, so our ears were intended for harmonious movements, and that these are in a manner sister sciences,[a] as the Pythagoreans assert and as we, Glaucon, agree.[b]

(iv.) *Logistic*

Scholium to Plato's *Charmides* 165 E

Logistic is the science that treats of numbered objects, not of numbers; it does not consider number in the true sense, but it works with 1 as unit and the numbered object as number, *e.g.*, it regards 3 as a triad and 10 as a decad, and applies the theorems of arithmetic to such cases. It is, then, logistic which treats on the one hand the problem called by Archimedes the cattle-problem,[c] and on the other hand *melite* and *phialite* numbers, the latter appertaining to bowls, the former to flocks[d]; in other types of problem too it has regard to the number of sensible bodies, treating them as absolute. Its subject-matter is everything that is numbered; its branches include the so-called Greek and Egyptian methods in multiplications and divisions, as well as the addi-

quantities connected by seven simple equations and subject to two other conditions. It involves the solution of a "Pellian" equation in numbers of fantastic size, and it is unlikely that Archimedes completed the solution. See vol. ii. pp. 202 ff.; T. L. Heath, *The Works of Archimedes*, pp. 319-326, and for a complete discussion, A. Amthor, *Zeitschrift für Math. u. Physik (Hist.-litt. Abtheilung)*, xxv. (1880), pp. 153-171, supplementing an article by B. Krumbiegel (pp. 121-136) on the authenticity of the problem.

[d] He should probably have said "apples".

σμοῖς καὶ μερισμοῖς, καὶ αἱ τῶν μορίων συγκεφαλαιώσεις καὶ διαιρέσεις, αἷς ἰχνεύει τὰ κατὰ τὴν ὕλην ἐμφωλευόμενα τῶν προβλημάτων τῇ περὶ τοὺς τριγώνους καὶ πολυγώνους πραγματείᾳ. τέλος δὲ αὐτῆς τὸ κοινωνικὸν ἐν βίῳ καὶ χρήσιμον ἐν συμβολαίοις, εἰ καὶ δοκεῖ περὶ τῶν αἰσθητῶν ὡς τελείων ἀποφαίνεσθαι.

(v.) *Later Classification*

Anatolius ap. Her., *Def.*, ed. Heiberg 164. 9-18

" Πόσα μέρη μαθηματικῆς;

" Τῆς μὲν τιμιωτέρας καὶ πρώτης ὁλοσχερέστερα μέρη δύο, ἀριθμητικὴ καὶ γεωμετρία, τῆς δὲ περὶ τὰ αἰσθητὰ ἀσχολουμένης ἕξ, λογιστική, γεωδαισία, ὀπτική, κανονική, μηχανική, ἀστρονομική. ὅτι οὔτε τὸ τακτικὸν καλούμενον οὔτε τὸ ἀρχιτεκτονικὸν οὔτε τὸ δημῶδες μουσικὸν ἢ τὸ περὶ τὰς φάσεις, ἀλλ' οὐδὲ τὸ ὁμωνύμως καλούμενον μηχανικόν, ὡς οἴονταί τινες, μέρη μαθηματικῆς εἰσι, προϊόντος δὲ τοῦ λόγου σαφῶς τε καὶ ἐμμεθόδως δείξομεν."

---

[a] *i.e.*, that which deals with non-sensible objects.

[b] Geminus, according to Proclus *in Eucl.* i. (ed. Friedlein 38. 8-12), gives the same classification, only in the order

tion and splitting up of fractions, whereby it explores the secrets lurking in the subject-matter of the problems by means of the theory of triangular and polygonal numbers. Its aim is to provide a common ground in the relations of life and to be useful in making contracts, but it appears to regard sensible objects as though they were absolute.

(v.) *Later Classification*

Anatolius, cited by Heron, *Definitions*, ed. Heiberg 164. 9-18

" How many branches of mathematics are there?

" There are two main branches of the prime and more honourable type of mathematics,[a] arithmetic and geometry; and there are six branches of that type of mathematics concerned with sensible objects, logistic, geodesy, optics, canonic, mechanics and astronomy.[b] That the so-called study of tactics and architecture and popular music and the study of [lunar] phases,[c] or even the mechanics so called homonymously,[d] are not branches of mathematics, as some think, we shall show clearly and methodically as the argument proceeds."

arithmetic, geometry, mechanics, astronomy, optics, geodesy, canonic, logistic. Geodesy means the practical measurement of surfaces and volumes: canonic is the theory of musical intervals; logistic is the art of calculation, as opposed to arithmetic, by which is meant what we should call the theory of numbers. Geminus proceeds to give an elaborate analysis of the various branches.

[c] According to Heiberg, this means " das Kalenderwesen."

[d] Heiberg interprets this as " die praktische Mechanik, die sich im Namen von der theoretischen nicht unterscheidet."

(b) Mathematics in Greek Education

Iambl. *De Vita Pythag.* 18. 89, ed. Deubner 52. 8-11

Λέγουσι δὲ οἱ Πυθαγόρειοι ἐξενηνέχθαι γεωμετρίαν οὕτως. ἀποβαλεῖν τινα τὴν οὐσίαν τῶν Πυθαγορείων· ὡς δὲ τοῦτο ἠτύχησε, δοθῆναι αὐτῷ χρηματίσασθαι ἀπὸ γεωμετρίας. ἐκαλεῖτο δὲ ἡ γεωμετρία πρὸς Πυθαγόρου ἱστορία.

Plat. *Leg.* vii. 817 E–820 D

ΑΘΗΝΑΙΟΣ ΞΕΝΟΣ. Ἔτι δὴ τοίνυν τοῖς ἐλευθέροις ἔστιν τρία μαθήματα, λογισμοὶ μὲν καὶ τὰ περὶ ἀριθμοὺς ἓν μάθημα, μετρητικὴ δὲ μήκους καὶ ἐπιπέδου καὶ βάθους ὡς ἓν αὖ δεύτερον, τρίτον δὲ τῆς τῶν ἄστρων περιόδου πρὸς ἄλληλα ὡς πέφυκεν πορεύεσθαι. ταῦτα δὲ σύμπαντα οὐχ ὡς ἀκριβείας ἐχόμενα δεῖ διαπονεῖν τοὺς πολλοὺς ἀλλά τινας ὀλίγους—οὓς δέ, προϊόντες ἐπὶ τῷ τέλει φράσομεν· οὕτω γὰρ πρέπον ἂν εἴη—τῷ πλήθει δέ, ὅσα αὐτῶν ἀναγκαῖα καί πως ὀρθότατα λέγεται μὴ ἐπίστασθαι μὲν τοῖς πολλοῖς αἰσχρόν, δι' ἀκριβείας δὲ ζητεῖν πάντα οὔτε ῥᾴδιον οὔτε τὸ παράπαν δυνατόν. . . .

Τοσάδε τοίνυν ἑκάστων χρὴ φάναι μανθάνειν δεῖν τοὺς ἐλευθέρους, ὅσα καὶ πάμπολυς ἐν Αἰγύπτῳ παίδων ὄχλος ἅμα γράμμασι μανθάνει. πρῶτον μὲν γὰρ περὶ λογισμοὺς ἀτεχνῶς παισὶν ἐξηυρημένα μαθήματα μετὰ παιδιᾶς τε καὶ ἡδονῆς μανθάνειν,

[a] Plato is thought to have redeemed this promise towards the end of the *Laws*, where he describes the composition of the Nocturnal Council, whose members are required to have considerable knowledge of mathematics.

[b] The Greek word is derived from the same root as the

(b) Mathematics in Greek Education

Iamblichus, *On the Pythagorean Life* 18. 89, ed. Deubner 52. 8-11

The Pythagoreans say that geometry was divulged in this manner. A certain Pythagorean lost his fortune; and when this befell him, he was permitted to make money from geometry. But geometry was called by Pythagoras "inquiry."

Plato, *Laws* vii. 817 E–820 D

Athenian Stranger. Then there are, of course, still three subjects for the freeborn to study. Calculations and the theory of numbers form one subject; the measurement of length and surface and depth make a second; and the third is the true relation of the movement of the stars one to another. To pursue all these studies thoroughly and with accuracy is a task not for the masses but for a select few—who these should be we shall say later towards the end of our argument, where it would be appropriate [a]—for the multitude it will be proper to learn so much of these studies as is necessary and so much as it can rightly be described a disgrace for the masses not to know, even though it would be hard, or altogether impossible, to pursue with precision all of those studies. . . .

Well then, the freeborn ought to learn as much of these things as a vast multitude of boys in Egypt learn along with their letters. First there should be calculations of a simple type devised for boys, which they should learn with amusement [b] and pleasure,

Greek word for "boy," and Plato is playing on the two words.

μήλων τέ τινων διανομαὶ καὶ στεφάνων πλείοσιν ἅμα καὶ ἐλάττοσιν ἁρμοττόντων ἀριθμῶν τῶν αὐτῶν, καὶ πυκτῶν καὶ παλαιστῶν ἐφεδρείας τε καὶ συλλήξεως ἐν μέρει καὶ ἐφεξῆς καὶ ὡς πεφύκασι γίγνεσθαι. καὶ δὴ καὶ παίζοντες, φιάλας ἅμα χρυσοῦ καὶ χαλκοῦ καὶ ἀργύρου καὶ τοιούτων τινῶν ἄλλων κεραννύντες, οἱ δὲ καὶ ὅλας πως διαδιδόντες, ὅπερ εἶπον, εἰς παιδιὰν ἐναρμόττοντες τὰς τῶν ἀναγκαίων ἀριθμῶν χρήσεις, ὠφελοῦσι τοὺς μανθάνοντας εἴς τε τὰς τῶν στρατοπέδων τάξεις καὶ ἀγωγὰς καὶ στρατείας καὶ εἰς οἰκονομίας αὖ, καὶ πάντως χρησιμωτέρους αὐτοὺς αὑτοῖς καὶ ἐγρηγορότας μᾶλλον τοὺς ἀνθρώπους ἀπεργάζονται· μετὰ δὲ ταῦτα ἐν ταῖς μετρήσεσιν, ὅσα ἔχει μήκη καὶ πλάτη καὶ βάθη, περὶ ἅπαντα ταῦτα ἐνοῦσάν τινα φύσει γελοίαν τε καὶ αἰσχρὰν ἄγνοιαν ἐν τοῖς ἀνθρώποις πᾶσιν, ταύτης ἀπαλλάτουσιν.

ΚΛΕΙΝΙΑΣ. Ποίαν δὴ καὶ τίνα λέγεις ταύτην;

ΑΘ. Ὦ φίλε Κλεινία, παντάπασί γε μὴν καὶ αὐτὸς ἀκούσας ὀψέ ποτε τὸ περὶ ταῦτα ἡμῶν πάθος ἐθαύμασα, καὶ ἔδοξέ μοι τοῦτο οὐκ ἀνθρώπινον ἀλλὰ ὑηνῶν τινων εἶναι μᾶλλον θρεμμάτων, ᾐσχύνθην τε οὐχ ὑπὲρ ἐμαυτοῦ μόνον, ἀλλὰ καὶ ὑπὲρ ἁπάντων τῶν Ἑλλήνων.

---

[a] Heath (*H.G.M.* i. 20 n. 1) first satisfactorily explained the construction of this sentence.

[b] The Athenian Stranger, generally taken to mean Plato

such as distributions of apples and crowns wherein the same numbers are divided among more or fewer, or distributions of the competitors in boxing and wrestling matches by the method of byes and drawings, or by taking them in consecutive order, or in any of the usual ways.[a] Again, the boys should play with bowls containing gold, bronze, silver and the like mixed together, or the bowls may be distributed as wholes. For, as I was saying, to incorporate in the pupils' play the elementary applications of arithmetic will be of advantage to them later in the disposition of armies, in marches and in campaigns, as well as in household management, and will make them altogether more useful to themselves and more awake. After these things there should be measurements of objects having length, breadth and depth, whereby they would free themselves from that ridiculous and shameful ignorance on all these topics which is the natural condition of all men.

Cleinias. And in what, pray, does this ignorance consist?

Athenian Stranger. My dear Cleinias, when I heard, somewhat belatedly, of our condition in this matter,[b] I also was astonished; such ignorance seemed to me worthy, not of human beings, but of swinish creatures, and I felt ashamed, not for myself alone, but for all the Greeks.

himself, proceeds to explain at length that he is referring to the problem of incommensurability. The Greek (ἀκούσας ὀψέ ποτε) could mean that he had only lately heard either of incommensurability itself or of the prevalent Greek ignorance about incommensurability. A. E. Taylor comments that in view of references to incommensurability in quite early dialogues it seems better to take the words in the latter sense.

ΚΛ. Τοῦ πέρι; λέγ' ὅτι καὶ φῄς, ὦ ξένε.

ΑΘ. Λέγω δή· μᾶλλον δὲ ἐρωτῶν σοι δείξω. καί μοι σμικρὸν ἀπόκριναι· γιγνώσκεις που μῆκος;

ΚΛ. Τί μήν;

ΑΘ. Τί δέ; πλάτος;

ΚΛ. Πάντως.

ΑΘ. Ἦ καὶ ταῦτα ὅτι δύ' ἐστόν, καὶ τρίτον τούτων βάθος;

ΚΛ. Πῶς γὰρ οὔ;

ΑΘ. Ἆρ' οὖν οὐ δοκεῖ σοι ταῦτα εἶναι πάντα μετρητὰ πρὸς ἄλληλα;

ΚΛ. Ναί.

ΑΘ. Μῆκός τε οἶμαι πρὸς μῆκος, καὶ πλάτος πρὸς πλάτος, καὶ βάθος ὡσαύτως δυνατὸν εἶναι μετρεῖν φύσει.

ΚΛ. Σφόδρα γε.

ΑΘ. Εἰ δ' ἔστι μήτε σφόδρα μήτε ἠρέμα δυνατὰ ἔνια, ἀλλὰ τὰ μέν, τὰ δὲ μή, σὺ δὲ πάντα ἡγῇ, πῶς οἴει πρὸς ταῦτα διακεῖσθαι;

ΚΛ. Δῆλον ὅτι φαύλως.

ΑΘ. Τί δ' αὖ μῆκός τε καὶ πλάτος πρὸς βάθος, ἢ πλάτος τε καὶ μῆκος πρὸς ἄλληλα; ἆρ' οὐ διανοούμεθα περὶ ταῦτα οὕτως Ἕλληνες πάντες, ὡς δυνατά ἐστι μετρεῖσθαι πρὸς ἄλληλα ἁμῶς γέ πως;

ΚΛ. Παντάπασι μὲν οὖν.

ΑΘ. Εἰ δ' ἔστιν αὖ μηδαμῶς μηδαμῇ δυνατά, πάντες δ', ὅπερ εἶπον, Ἕλληνες διανοούμεθα ὡς δυνατά, μῶν οὐκ ἄξιον ὑπὲρ πάντων αἰσχυνθέντα εἰπεῖν πρὸς αὐτούς· "Ὦ βέλτιστοι τῶν Ἑλλήνων, ἓν ἐκείνων τοῦτ' ἐστὶν ὧν ἔφαμεν αἰσχρὸν μὲν

Clein. Why? Please explain, sir, what you are saying.

Ath. I will indeed do so; or rather I will make it plain to you by asking questions. Pray, answer me one little thing; you know what is meant by *line*?

Clein. Of course.

Ath. And again by *surface*?

Clein. Certainly.

Ath. And you know that these are two distinct things, and that *volume* is a third distinct from them?

Clein. Even so.

Ath. Now does not it appear to you that they are all commensurable one with another?

Clein. Yes.

Ath. I mean, that line is in its nature measurable by line, and surface by surface, and similarly with volume.

Clein. Most assuredly.

Ath. But suppose this cannot be said of some of them, neither with more assurance nor with less, but is in some cases true, in others not, and suppose you think it true in all cases; what you do think of your state of mind in this matter?

Clein. Clearly, that it is unsatisfactory.

Ath. Again, what of the relations of line and surface to volume, or of surface and line one to another; do not all we Greeks imagine that they are commensurable in some way or other?

Clein. We do indeed.

Ath. Then if this is absolutely impossible, though all we Greeks, as I was saying, imagine it possible, are we not bound to blush for them all as we say to them, "Worthy Greeks, this is one of the things of which we said that ignorance is a disgrace and that

γεγονέναι τὸ μὴ ἐπίστασθαι, τὸ δ' ἐπίστασθαι τἀναγκαῖα οὐδὲν πάνυ καλόν;"

ΚΛ. Πῶς δ' οὔ;

ΑΘ. Καὶ πρὸς τούτοις γε ἄλλα ἔστιν τούτων συγγενῆ, ἐν οἷς αὖ πολλὰ ἁμαρτήματα ἐκείνων ἀδελφὰ ἡμῖν ἐγγίγνεται τῶν ἁμαρτημάτων.

ΚΛ. Ποῖα δή;

ΑΘ. Τὰ τῶν μετρητῶν τε καὶ ἀμέτρων πρὸς ἄλληλα ἥτινι φύσει γέγονεν· ταῦτα γὰρ δὴ σκοποῦντα διαγιγνώσκειν ἀναγκαῖον ἢ παντάπασιν εἶναι φαῦλον, προβάλλοντά τε ἀλλήλοις ἀεί, διατριβὴν τῆς πεττείας πολὺ χαριεστέραν πρεσβυτῶν διατρίβοντα, φιλονικεῖν ἐν ταῖς τούτων ἀξίαισι σχολαῖς.

ΚΛ. Ἴσως· ἔοικεν γοῦν ἥ τε πεττεία καὶ ταῦτα ἀλλήλων τὰ μαθήματα οὐ πάμπολυ κεχωρίσθαι.

Isoc. *Panathenaicus* 26-28, 238 B-D

Τῆς μὲν οὖν παιδείας τῆς ὑπὸ τῶν προγόνων καταλειφθείσης τοσούτου δέω καταφρονεῖν, ὥστε καὶ τὴν ἐφ' ἡμῶν κατασταθεῖσαν ἐπαινῶ, λέγω δὲ

---

[a] Plato is probably censuring a belief that if two squares are commensurable, their sides are also commensurable; and if two cubes are commensurable, their surfaces and sides are also commensurable. The discovery that this is not necessarily so would arise in such problems as that propounded in *Meno* 82 B—85 B (doubling of a square) and in the duplication of the cube (see *infra*, pp. 256-309). The only difficulty is that commensurability is not always impossible (μηδαμῶς μηδαμῇ δυνατά). A belief that areas and volumes can be expressed in linear measure would meet this stipulation, but it seems too elementary to call for elaborate refutation by Plato.

to know such necessary matters is no great achievement"?[a]

CLEIN. Certainly.

ATH. In addition to these, there are other related points, which often give rise to errors akin to those lately mentioned.

CLEIN. What kind of errors do you mean?

ATH. The real nature of commensurables and incommensurables towards one another.[b] A man must be able to distinguish them on examination, or must be a very poor creature. We should continually put such problems to each other—it would be a much more elegant occupation for old people than draughts—and give our love of victory an outlet in pastimes worthy of us.

CLEIN. Perhaps so; it would seem that draughts and these studies are not so widely separated.

Isocrates, *Panegyric of Athens* 26-28, 238 B-D[c]

So far from despising the education handed down by our ancestors, I even approve that established in

[b] According to A. E. Taylor, this means that "behind the more special problems of the commensurability of specific areas and volumes there lies the problem of constructing a general 'theory of incommensurables.'" He calls in the evidence of Epinomis, 990 B—991 B, for which see *infra*, pp. 400-405. For further references to the problem see *infra*, pp. 110-111, 214-215.

[c] Isocrates began this last of his orations in his ninety-fourth year and it was published in his ninety-eighth. He expresses similar sentiments about mathematics in *Antidosis* §§ 261-268; see also Xenophon, *Memorabilia* iv. 7. 2 ff. Heath's dry comment (*H.G.M.* i. 22) is: "It would appear therefore that, notwithstanding the influence of Plato, the attitude of cultivated people in general towards mathematics was not different in Plato's time from what it is to-day."

τήν τε γεωμετρίαν καὶ τὴν ἀστρολογίαν καὶ τοὺς διαλόγους τοὺς ἐριστικοὺς καλουμένους, οἷς οἱ μὲν νεώτεροι μᾶλλον χαίρουσι τοῦ δέοντος, τῶν δὲ πρεσβυτέρων οὐδεὶς ἔστιν, ὅστις ἂν ἀνεκτοὺς αὐτοὺς εἶναι φήσειεν.

Ἀλλ' ὅμως ἐγὼ τοῖς ὡρμημένοις ἐπὶ ταῦτα παρακελεύομαι πονεῖν καὶ προσέχειν τὸν νοῦν ἅπασι τούτοις, λέγων, ὡς εἰ καὶ μηδὲν ἄλλο δύναται τὰ μαθήματα ταῦτα ποιεῖν ἀγαθόν, ἀλλ' οὖν ἀποτρέπει γε τοὺς νεωτέρους πολλῶν ἄλλων ἁμαρτημάτων. τοῖς μὲν οὖν τηλικούτοις οὐδέποτ' ἂν εὑρεθῆναι νομίζω διατριβὰς ὠφελιμωτέρας τούτων οὐδὲ μᾶλλον πρεπούσας· τοῖς δὲ πρεσβυτέροις καὶ τοῖς εἰς ἄνδρας δεδοκιμασμένοις οὐκέτι φημὶ τὰς μελέτας ταύτας ἁρμόττειν. ὁρῶ γὰρ ἐνίους τῶν ἐπὶ τοῖς μαθήμασι τούτοις οὕτως ἀπηκριβωμένων ὥστε καὶ τοὺς ἄλλους διδάσκειν, οὔτ' εὐκαίρως ταῖς ἐπιστήμαις αἷς ἔχουσι χρωμένους, ἔν τε ταῖς ἄλλαις πραγματείαις ταῖς περὶ τὸν βίον ἀφρονεστέρους ὄντας τῶν μαθητῶν· ὀκνῶ γὰρ εἰπεῖν τῶν οἰκετῶν.

(c) Practical Calculation

(i.) *Enumeration by Fingers*

Aristot. *Prob.* xv. 3, 910 b 23–911 a 1

Διὰ τί πάντες ἄνθρωποι, καὶ βάρβαροι καὶ Ἕλληνες, εἰς τὰ δέκα καταριθμοῦσι, καὶ οὐκ εἰς ἄλλον ἀριθμόν, οἷον β̄, γ̄, δ̄, ε̄, εἶτα πάλιν ἐπαναδιπλοῦσιν, ἓν πέντε, δύο πέντε, ὥσπερ ἕνδεκα, δώδεκα; . . . ἢ ὅτι πάντες ὑπῆρξαν ἄνθρωποι ἔχοντες δέκα δακτύλους; οἷον οὖν ψήφους ἔχοντες

our own times—I mean geometry, astronomy, and the so-called eristic dialogues, in which our young men delight more than they ought, though there is not one of the older men who would pronounce them tolerable.

Nevertheless I urge those who are inclined to these disciplines to work hard and apply their mind to all of them, saying that even if these studies can do no other good, they at least keep the young out of many other things that are harmful. Indeed, for those who are at this age I maintain that no more helpful or fitting occupations can be found; but for those who are older and those admitted to man's estate I assert that these disciplines are no longer suitable. For I notice that some of those who have become so versed in these studies as to teach others fail to use opportunely the sciences they know, while in the other activities of life they are more unpractical than their pupils—I shrink from saying than their servants.

(c) Practical Calculation

(i.) *Enumeration by Fingers*

Aristotle, *Problems* xv. 3, 910 b 23–911 a 1

Why do all men, both barbarians and Greeks, count up to ten and not up to any other number, such as 2, 3, 4 or 5, whence they would start again, saying, for example, one *plus* five, two *plus* five, just as they say one *plus* ten, two *plus* ten?[a] . . . Is it that all men were born with ten fingers? Having the

[a] The Greek words for 11 and 12 mean literally *one-ten*, *two-ten*.

τοῦ οἰκείου ἀριθμοῦ, τούτῳ τῷ πλήθει καὶ τὰ ἄλλα ἀριθμοῦσιν.

Nicolas Rhabdas, ed. Tannery, *Notices et extraits des manuscrits de la Bibliothèque Nationale*, vol. xxxii. pt. 1, pp. 146-152

Ἔκφασις τοῦ δακτυλικοῦ μέτρου

Ἐν δὲ ταῖς χερσὶ καθέξεις τοὺς ἀριθμοὺς οὕτως· καὶ ἐν μὲν τῇ λαιᾷ, ὀφείλεις ἀεὶ τοὺς μοναδικοὺς καὶ δεκαδικοὺς κρατεῖν ἀριθμούς, ἐν δὲ τῇ δεξιᾷ τοὺς ἑκατονταδικοὺς καὶ χιλιονταδικούς, τοὺς δὲ ἐπέκεινα τούτων χαράττειν ἔν τινι· οὐ γὰρ ἔχεις ὅπως καθέξεις ἐν ταῖς χερσί.

Συστελλομένου τοῦ πρώτου καὶ μικροῦ δακτύλου, τοῦ μύωπος καλουμένου, τῶν δὲ τεσσάρων ἐκτεταμένων καὶ ἱσταμένων ὀρθίων, κατέχεις ἐν μὲν τῇ ἀριστερᾷ χειρὶ μονάδα μίαν, ἐν δὲ τῇ δεξιᾷ χιλιοντάδα μίαν.

Καὶ πάλιν συστελλομένου καὶ τούτου καὶ τοῦ μετ᾽ αὐτὸν δευτέρου δακτύλου, τοῦ παραμέσου καὶ ἐπιβάτου καλουμένου, τῶν δὲ λοιπῶν τριῶν ὡς ἔφημεν ἡπλωμένων, κρατεῖς ἐν μὲν τῇ εὐωνύμῳ δύο, ἐν δὲ τῇ δεξιᾷ δισχίλια.

Τοῦ δ᾽ αὖ τρίτου συστελλομένου, ἤτοι τοῦ σφακέλου καὶ μέσου, κειμένων καὶ τῶν ἑτέρων δύο, τῶν

---

[a] The word πεμπάζειν ("to five"), used by Homer (*Od.* iv. 412) in the sense "to count," would appear to be a relic of a quinary system of reckoning. The Greek χείρ, like the Latin *manus*, is used to denote "a number" of men, *e.g.*, Herodotus vii. 157, viii. 140 ; Thucydides iii. 96.

[b] Nicolas Artavasdas of Smyrna, called Rhabdas, lived in the fourteenth century A.D. He is the author of two letters

equivalent of pebbles to the number of their own fingers, they came to use this number for counting everything else as well.[a]

Nicolas Rhabdas,[b] ed. Tannery, *Notices et extraits des manuscrits de la Bibliothèque Nationale*, vol. xxxii. pt. 1, pp. 146-152

Exposition of finger-notation[c]

This is how numbers are represented on the hands: The left hand is always used for the units and tens, and the right hand for the hundreds and thousands, while beyond that some form of characters must be used, for the hands are not sufficient.

Closing the first finger—the little one, called *myope*—and keeping the other four stretched out straight, you have on the left hand 1 and on the right hand 1000.[d]

Again, closing this finger together with that next after it—the second, called *next the middle* and *epibate*—and keeping the remaining three fingers open, as we said, you have on the left hand 2 and on the right hand 2000.

Once more, closing the third finger—called *sphakelos* and *middle*—and keeping the other two as

edited by Tannery, of which the second can be dated to the year 1341 by a calculation of Easter. He edited the arithmetical manual of the monk Maximus Planudes.

[c] A similar system is explained by the Venerable Bede, *De temporum ratione*, c. i., "De computo vel loquela digitorum." He implies that St. Jerome (*ob.* A.D. 420) was also acquainted with the system.

[d] In the Greek the numerals are sometimes written in full, sometimes in the alphabetic notation, for which see *infra*, p. 43.

δὲ λοιπῶν δύο ἐκτεταμένων, τοῦ λιχανοῦ λέγω καὶ τοῦ ἀντίχειρος, εἰσὶν ἅπερ κρατεῖς ἐν μὲν τῇ λαιᾷ, γ̄, ἐν δὲ τῇ δεξιᾷ, ,γ̄.

Πάλιν συστελλομένων τῶν δύο, τοῦ μέσου καὶ παραμέσου, ἤγουν τοῦ δευτέρου καὶ τρίτου, καὶ τῶν ἄλλων ὄντων ἐξηπλωμένων, τοῦ ἀντίχειρος λέγω, τοῦ λιχανοῦ καὶ τοῦ μύωπος, εἰσὶν ἅπερ κρατεῖς ἐν μὲν τῇ λαιᾷ, δ̄, ἐν δὲ τῇ δεξιᾷ, ,δ̄.

Πάλιν τοῦ τρίτου, τοῦ καὶ μέσου, συνεσταλμένου, καὶ τῶν λοιπῶν τεσσάρων ἐκτεταμένων, δηλοῦσιν ἅπερ κρατεῖς ⟨ἐν μὲν τῇ λαιᾷ⟩[1] ε̄, ἐν δὲ τῇ δεξιᾷ, ,ε̄.

Τοῦ ἐπιβάτου πάλιν, τοῦ καὶ δευτέρου, συνεσταλμένου καὶ τῶν λοιπῶν ⟨τεσσάρων⟩[2] ἡπλωμένων, κρατεῖς ἐν μὲν τῇ εὐωνύμῳ ϛ̄, ἐν δὲ τῇ ἑτέρᾳ ,ϛ̄.

Τοῦ μύωπος πάλιν, τοῦ καὶ πρώτου, ἐκτεταμένου καὶ τῇ παλάμῃ προσψαύοντος, τῶν δὲ λοιπῶν ἱσταμένων ὀρθίως, εἰσὶν ἅπερ κατέχεις, ζ̄, ἐν δὲ τῇ ἄλλῃ, ,ζ̄.

Τοῦ δευτέρου πάλιν, τοῦ καὶ παραμέσου, ὁμοίως ἐκτεταμένου καὶ κλίνοντος ἄχρις οὗ τῇ κνάθῳ τελείως προσεγγίσῃ, τῶν δὲ λοιπῶν τριῶν, τοῦ τρίτου, τοῦ τετάρτου καὶ τοῦ πέμπτου, ὡς προείρηται ἱσταμένων ὀρθίων, τὸ γενόμενον σχῆμα ἐν μὲν τῇ λαιᾷ δηλοῖ η̄, ἐν δὲ τῇ δεξιᾷ ,η̄.

Οὕτως οὖν καὶ τοῦ τρίτου γενομένου, κειμένων καὶ τῶν ἄλλων δύο, τοῦ πρώτου καὶ δευτέρου, κατὰ τὸ αὐτὸ σχῆμα, ἐν μὲν τῇ ἀριστερᾷ δηλοῦσιν θ̄, ἐν δὲ τῇ ἄλλῃ ,θ̄.

Πάλιν τοῦ ἀντίχειρος ἡπλωμένου, οὐχὶ δ' ὑπερ-

[1] ἐν . . . λαιᾷ add. Morel.
[2] τεσσάρων add. Tannery.

before, with the remaining two held out straight—I mean the forefinger[a] and thumb[b]—you have on the left hand 3 and on the right hand 3000.

Again, closing the two fingers called *middle* and *next the middle*, that is, the second and third, and keeping the others open—I mean the thumb and forefinger and that called *myope*, you have on the left hand 4 and on the right hand 4000.

Again, closing the third finger—the *middle*—and keeping the remaining four straight, the fingers will represent on the left hand 5 and on the right hand 5000.

Closing, again, the *epibate* finger—the second—and keeping the remaining four open, you have on the left hand 6 and on the other 6000.

Again, by extending the finger called *myope*—the first—so as to touch the palm, and keeping the others stretched out straight, you have 7 and on the other hand 7000.

If the second finger—that called *next the middle*—is extended in a similar manner and bent until it nearly touches the hollow of the hand, while the remaining three fingers—the third, fourth and fifth—are stretched out straight as aforesaid, the resulting figure will represent on the left hand 8 and on the right hand 8000.

If the third finger also is bent in this manner, the other two—the first and second—remaining as before, the fingers will represent on the left hand 9 and on the other 9000.

Again, if the thumb is kept open, not raised verti-

[a] The Greek word means literally the "licking" finger.
[b] The Greek word means literally "that which is opposite" *sc.* the four fingers.

αιρομένου, ἀλλὰ πλαγίως πως, καὶ τοῦ λιχανοῦ ὑποκλινομένου ἄχρις ἂν τῷ τοῦ ἀντίχειρος προτέρῳ ἄρθρῳ συμπέσῃ, ἕως ἂν γένηται σίγματος σχῆμα, τῶν δὲ λοιπῶν τριῶν φυσικῶς ἡπλωμένων καὶ μὴ χωριζομένων ἀπ' ἀλλήλων, ἀλλὰ συνημμένων, τὸ τοιοῦτον ἐν μὲν τῇ εὐωνύμῳ χειρὶ σημαίνει δέκα, ἐν δὲ τῇ δεξιᾷ ρ̄.[a]

(ii.) *The Abacus*[b]

Herod. ii. 36. 4

Γράμματα γράφουσι καὶ λογίζονται ψήφοισι Ἕλληνες μὲν ἀπὸ τῶν ἀριστερῶν ἐπὶ τὰ δεξιὰ φέροντες τὴν χεῖρα, Αἰγύπτιοι δὲ ἀπὸ τῶν δεξιῶν ἐπὶ τὰ ἀριστερά· καὶ ποιεῦντες ταῦτα αὐτοὶ μέν φασι ἐπὶ δεξιὰ ποιέειν, Ἕλληνας δὲ ἐπ' ἀριστερά.

---

[a] It is perhaps unnecessary to follow this trifle to its end. Rhabdas proceeds to show how the tens from 20 to 90, and the hundreds from 200 to 900, can be represented in similar manner. Details are given in Heath, *H.G.M.* ii. 552.

I have not found it possible to give a satisfactory rendering of Rhabdas's names for the fingers. Possibly μύωψ should be translated *spur* (though this seems a more natural name for the thumb than the first finger) and ἐπιβάτης *rider*; σφάκελος (σφάκελλος in the mss.) can mean spasms or convulsions, and Mr. Colin Roberts tentatively suggests (to my mind convincingly) that the middle finger is so called because it is joined with the thumb in cracking the fingers.

[b] The only ancient abaci which have been preserved and can definitely be identified as such are Roman. It is disputed whether the famous Salaminian table, discovered by

cally but somewhat aslant, and the forefinger is bent until it touches the first joint of the thumb, so that they resemble the letter σ, while the remaining three fingers are kept open in their natural position and not separated from each other but kept together, the figure so formed will signify on the left hand 10 and on the right hand 100.[a]

(ii.) *The Abacus*[b]

Herodotus ii. 36. 4

In writing and in reckoning with pebbles the Greeks move the hand from left to right, but the Egyptians from right to left[c]; in so doing they maintain that they move the hand to the right, and that it is the Greeks who move to the left.

Rangabé and described by him in 1846 (*Revue archéologique* iii.), is an abacus or a game-board; the table now lies in the Epigraphical Museum at Athens and is described and illustrated by Kubitschek (*Wiener numismatische Zeitschrift*, xxxi., 1899, pp. 393-398, with Plate xxiv.), Nagl (*Abhandlungen zur Geschichte der Mathematik*, ix., 1899, plate after p. 357) and Heath, *H.G.M.* i. 49-51. The essence of the Greek abacus, like the Roman, was an arrangement of the columns to denote different denominations, *e.g.*, in the case of the decimal system units, tens, hundreds, and thousands. The number of units in each denomination was shown by pebbles. When the pebbles collected in one column became sufficient to form one or more units of the next highest denomination, they were withdrawn and the proper number of pebbles substituted in the higher column.

[c] This implies that the columns were vertical.

Diog. Laert. i. 59

Ἔλεγε δὲ τοὺς παρὰ τοῖς τυράννοις δυναμένους παραπλησίους εἶναι ταῖς ψήφοις ταῖς ἐπὶ τῶν λογισμῶν. καὶ γὰρ ἐκείνων ἑκάστην ποτὲ μὲν πλείω σημαίνειν, ποτὲ δὲ ἧττω· καὶ τούτων τοὺς τυράννους ποτὲ μὲν ἕκαστον μέγαν ἄγειν καὶ λαμπρόν, ποτὲ δὲ ἄτιμον.

Polyb. *Histor.* v. 26. 13

Ὄντως γάρ εἰσιν οὗτοι παραπλήσιοι ταῖς ἐπὶ τῶν ἀβακίων ψήφοις· ἐκεῖναί τε γὰρ κατὰ τὴν τοῦ ψηφίζοντος βούλησιν ἄρτι χαλκοῦν καὶ παραυτίκα τάλαντον ἰσχύουσιν, οἵ τε περὶ τὰς αὐλὰς κατὰ τὸ τοῦ βασιλέως νεῦμα μακάριοι καὶ παρὰ πόδας ἐλεεινοὶ γίνονται.

Diogenes Laertius i. 59

He [Solon] used to say that men who surrounded tyrants were like the pebbles used in calculations; for just as each pebble stood now for more, now for less, so the tyrants would treat each of their courtiers now as great and famous, now as of no account.

Polybius, *History* v. 26. 13

These men are really like the pebbles on reckoning-boards. For the pebbles, according to the will of the reckoner, have the value now of an eighth of an obol, and the next moment of a talent [a]; while courtiers, at the nod of the king, are now happy, and the next moment lying piteously at his feet.

[a] In the Salaminian table (see *supra*, p. 34 n. *b*) the extreme denominations on one side are actually the talent and the χαλκοῦς ($\frac{1}{8}$ obol).

II. ARITHMETICAL NOTATION AND THE CHIEF ARITHMETICAL OPERATIONS

II. ARITHMETICAL NOTATION AND THE CHIEF ARITHMETICAL OPERATIONS

(a) English Notes and Examples

From earliest times the Greeks followed the decimal system of enumeration. At first, no doubt, the words for the different numbers were written out in full, and many inscriptions bear witness to this practice. But the development of trade and of mathematical interests would soon have caused the Greeks to search for some more convenient symbolic method of representing numbers. The first system of symbols devised for this purpose is sometimes known as the Attic system, owing to the prevalence of the signs in Attic inscriptions. In it I represents the unit, and may be repeated up to four times. There are only five other distinct symbols, each being the first letter of the word representing a number. They are

Γ (the first letter of πέντε)	= 5
Δ (δέκα)	= 10
Η (ἕκατον)	= 100
Χ (χίλιοι)	= 1000
Μ (μύριοι)	= 10000

Like I, each of these signs may be repeated up to

four times. Four other symbols are formed by compounding two of the simple signs.

𐅄 (Γ and Δ) = 50
𐅅 (Γ and H) = 500
𐅆 (Γ and X) = 5000
𐅇 (Γ and M) = 50000

By combinations of these signs it is possible to represent any number from 1 to 50000. For example, 𐅆XHHHΔΔΓΙΙΙΙ = 6329.

Notwithstanding the opinion of Cantor,[a] there is very little to be said for this cumbrous notation. A second system devised by the Greeks made use of the letters of the alphabet, with three added letters, as numerals. It is not certain when this system came into use,[b] but it had completely superseded the older system long before the time of the writers with whom we shall be concerned, and for the purposes of this book it is the only system which need be noticed. In it an alphabet of 27 letters is used: the first nine letters represent the units from 1 to 9, the second nine represent the tens from 10 to 90, and the third nine represent the hundreds from 100 to 900. To show that a numeral is indicated, a horizontal stroke

[a] *Vorlesungen über Geschichte der Mathematik*, i³, p. 129.
[b] For a full consideration of the date given by Larfeld (end of eighth century B.C.) and that given by Keil (550–425 B.C.), see Heath, *H.G.M.* i. 33-34.

is generally placed above the letter in cursive writing, as in the following scheme [a]

$\bar{\alpha}=1$	$\bar{\iota}=10$	$\bar{\rho}=100$
$\bar{\beta}=2$	$\bar{\kappa}=20$	$\bar{\sigma}=200$
$\bar{\gamma}=3$	$\bar{\lambda}=30$	$\bar{\tau}=300$
$\bar{\delta}=4$	$\bar{\mu}=40$	$\bar{\upsilon}=400$
$\bar{\epsilon}=5$	$\bar{\nu}=50$	$\bar{\phi}=500$
ϛ̄ $=6$	$\bar{\xi}=60$	$\bar{\chi}=600$
$\bar{\zeta}=7$	$\bar{o}=70$	$\bar{\psi}=700$
$\bar{\eta}=8$	$\bar{\pi}=80$	$\bar{\omega}=800$
$\bar{\theta}=9$	ϟ̄ $=90$	ϡ̄ $=900$

The horizontal stroke is often omitted for convenience in printed texts.

In this system there are three letters ϛ (Stigma, a form of the digamma), ϟ or ϙ (Koppa) and ϡ (Sampi) which had been taken over by the Greeks from the Phoenician alphabet but had dropped out of literary use. As there is no record of this alphabet of 27 letters in this order being in use at any time, it seems to have been deliberately framed by someone for the purposes of mathematics.[b] Though more concise than the Attic system, it suffers from the disadvantage of giving no indication of place-value; the connexion between ϵ, $\bar{\nu}$ and $\bar{\phi}$, for example, does not leap to the eye as in the Arabic notation 5, 50, 500.

[a] In some texts the method of indicating that a letter stands for a numeral is an accent placed above the letter and to the right, in the following manner:

$$\alpha'=1,\ \iota'=10,\ \rho'=100.$$

A double accent is used to indicate submultiples, *e.g.*,

$$\gamma''=\tfrac{1}{3},\ \lambda''=\tfrac{1}{30},\ \tau''=\tfrac{1}{300}.$$

[b] Gow, *A Short History of Greek Mathematics*, pp. 45-46.

Opinions differ greatly on the facility with which it could be used, but the balance of opinion is in favour of the view that it was an obstacle to the development of arithmetic by the Greeks.

By combination of these letters, it is possible to represent any number from 1 to 999. Thus $\overline{\rho\nu\gamma}=153$. For the thousands from 1000 to 9000 the letters α to θ are used again with a distinguishing mark, generally a stroke subscribed to the letter a little to the left, in addition to the horizontal stroke above the letter.

Thus $\quad {}_{,}\bar{\alpha}=1000,\ {}_{,}\bar{\beta}=2000,\ .\ .\ .\ {}_{,}\bar{\theta}=9000.$

For tens of thousands the sign M is used, generally with the number of myriads written above it.

Thus $\quad \overset{\alpha}{\mathrm{M}}=10000,\ \overset{\beta}{\mathrm{M}}=20000$, and so on (Eutocius).

Another method is to use the sign M or $\overset{\Upsilon}{\mathrm{M}}$ for the myriad and to put the number of myriads after it, separated by a dot from the thousands.

Thus

$\overset{\Upsilon}{\mathrm{M}}\overline{\rho\delta}.\ \overline{{}_{,}\eta\phi o\varsigma}=1048576$ (Diophantus vi. 22, ed. Tannery 446. 11).

In a third method the symbol M is not used, but the symbol representing the number of myriads has two dots placed over it.

Thus

$\ddot{\alpha}\overline{{}_{,}\eta\phi\text{Ϟ}\varsigma}=18596$ (Heron, *Geometrica* xvii. 33, ed. Heiberg 348. 35).

Heron commonly wrote the word *μυριάδες* in full.

To express still higher numbers, powers of myriads were used. Apollonius and Archimedes invented

systems of "tetrads" and "octads" respectively to indicate powers of 10000 and 100000000.

There was no single Greek system for representing fractions. With submultiples, the orthodox method was to write the letter for the corresponding number with an accent instead of a horizontal dash, *e.g.*, $\delta' = \frac{1}{4}$. There were special signs, ∠′ and C′, for $\frac{1}{2}$, and ω′ for $\frac{2}{3}$. The Greeks, like the Egyptians, tried to express ordinary proper fractions as the sum of two or more submultiples. Thus ∠′ $\delta' = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, ∠′ $\xi\delta' = \frac{1}{2} + \frac{1}{64} = \frac{33}{64}$ (Eutocius). There was a limit to what could be done in this way, and the Greeks devised several methods of representing ordinary proper fractions. The most convenient is that used by Diophantus, and occasionally by Heron. The numerator is written underneath the denominator, which is the reverse of our modern practice. Thus $\overset{\chi o\varsigma}{\iota\epsilon} = \frac{15}{676}$. A method commonly used in Heron's works was to write the denominator twice and with an accent, *e.g.*, $\bar{\delta}\ \zeta'\zeta' = \frac{4}{7}$, $\overline{\iota\beta}\ \zeta'\zeta' = \frac{12}{7}$. Sometimes the word λεπτά ("fractional parts") was added, *e.g.*, λεπτὰ να′ να′ $\overline{\lambda\epsilon} = \frac{35}{51}$. There is no fixed order of preference for numerator and denominator. In Aristarchus of Samos we find δύο με′ for $\frac{2}{45}$ and in Archimedes ι οα′ for $\frac{10}{71}$, where only the context will show that $10\frac{1}{71}$ is not intended.

Several fragments illustrating elementary mathematical operations have come to light among the Egyptian papyri.[a] The following tables (2nd cent. A.D.) show how fractions can be represented as sums of submultiples. The Greek is set out in columns. The

[a] I am indebted to Mr. Colin Roberts for drawing my attention to them.

first two columns give the numerator of the fraction to be split up. The denominator is not explicitly announced in the table, but it is implicit in the first line. Fractions are marked with signs like accents, usually but not always over every letter. The sign Δ for $\frac{1}{4}$ will be noted. Dots under letters indicate doubtful readings.

Michigan Papyri, No. 145, vol. iii. (*Humanistic Series*, vol. xl.) p. 36

I, ii

A Table of Twenty-thirds

της	α	κ'γ'		
[των	β]	ι'β'	σο'ς'	
[των	γ]	ι'	μ'ς'	ρ'ι'ε'
[των	δ	ς]'	ρ'λ'η'	
[των	ε	ς'	κ'γ]'	ρλ'[η']

Equivalent in Arabic Notation

$$\tfrac{1}{23} = \tfrac{1}{23}$$
$$\tfrac{2}{23} = \tfrac{1}{12} + \tfrac{1}{276}$$
$$\tfrac{3}{23} = \tfrac{1}{10} + \tfrac{1}{46} + \tfrac{1}{115}$$
$$\tfrac{4}{23} = \tfrac{1}{6} + \tfrac{1}{138}$$
$$\tfrac{5}{23} = \tfrac{1}{6} + \tfrac{1}{23} + \tfrac{1}{138}$$

ii

A Table of Twenty-ninths

των	ιβ	Δ	η'	κθ'	σλ'β'	
[των]	ιγ	γ'	ι'ε'	[κ'θ'	π']ζ'	υ[λ'ε']
[των]	ιδ	Δ	ε'	[ν']η'	ρις'	ρμ'ε'
[των]	ιε	∠	ν'η'			
[των	ι]ς		[∠	κ']θ'	ν'η'	
[των	ιζ	∠		ι'β']	τ'μ'η'	

Equivalent in Arabic Notation

$$\tfrac{12}{29}=\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{29}+\tfrac{1}{232}$$
$$\tfrac{13}{29}=\tfrac{1}{3}+\tfrac{1}{15}+\tfrac{1}{29}+\tfrac{1}{87}+\tfrac{1}{435}$$
$$\tfrac{14}{29}=\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{58}+\tfrac{1}{116}+\tfrac{1}{145}$$
$$\tfrac{15}{29}=\tfrac{1}{2}+\tfrac{1}{58}$$
$$\tfrac{16}{29}=\tfrac{1}{2}+\tfrac{1}{29}+\tfrac{1}{58}$$
$$\tfrac{17}{29}=\tfrac{1}{2}+\tfrac{1}{12}+\tfrac{1}{348}$$

The Greeks had no sign corresponding to 0, and never rose to the conception of 0 as a number.[a] Having no need of a sign to indicate decimal position, they wrote such a number as 1007 in only two letters— $\overline{,\alpha\zeta}$.

By means of these devices the Greeks had a complete system of enumeration. Here are a few examples of complicated numbers taken from Eutocius:

$$\overset{\rho\lambda\zeta}{\mathrm{M}}\ \overline{,\gamma\sampi\mu\gamma}\ \angle'\ \xi\delta' = 1373943\tfrac{1}{2}\tfrac{1}{64} = 1373943\tfrac{33}{64}.$$

$$\overset{\phi\mu\zeta}{\mathrm{M}}\ \overline{,\beta\koppa}\ \angle'\ \iota\varsigma' = 5472090\tfrac{1}{2}\tfrac{1}{16} = 5472090\tfrac{9}{16}.$$

With these symbols the Greeks conducted the chief mathematical operations in much the same manner, and with much the same facility, as we do. The following is an example of multiplication from

[a] In his sexagesimal notation, Ptolemy used the symbol O to stand for οὐδεμία μοῖρα or οὐδὲν ἑξηκοστόν. The diverse views which have been held on this symbol from the time of Delambre are summed up by Loria (*Le scienze esatte nell' antica Grecia*, p. 761) in the words: "In base ai documenti scoperti e decifrati sino ad oggi, siamo autorizzati a negare che i Greci usassero lo zero nel senso e nel modo in cui lo adoperiamo noi."

Eutocius's commentary on Archimedes' *Measurement of a Circle* (Archim., ed. Heiberg iii. 242):

	$\overline{\rho\nu\gamma}$		153		
ἐπὶ	$\overline{\rho\nu\gamma}$		×153		
	$\overset{\alpha}{\mathrm{M}}\ ,\overline{\epsilon\tau}$		15300		
	$,\epsilon\ ,\overline{\beta\phi}\ \overline{\rho\nu}$		5000	2500	150
	$\tau\ \overline{\rho\nu\theta}$			300	159
ὁμοῦ	$\overset{\beta}{\mathrm{M}}\ ,\overline{\gamma\upsilon\theta}$	Total	23409		

The operation, it will be noticed, is split up into a number of simple operations. 153 is first multiplied by 100, then 100, 50 and 3 are separately multiplied by 50, and lastly 100 and 53 are separately multiplied by 3. The products are finally all added together to make the total of 23409.

Only one example of long division fully worked out survives in the whole of the extant corpus of Greek mathematical writings—in Theon's *Commentary on the Syntaxis of Ptolemy*. The same work contains an example of the extraction of a square root. Both passages will be reproduced, but as the notation is sexagesimal a few words of explanation are necessary.

The sexagesimal notation had its origin among the Babylonians and was used by the Greeks in astronomical calculations. It appears fully developed in the *Syntaxis* of Ptolemy and the *Commentaries* of Theon and Pappus.[a] In this system the circumference of a

[a] Theon of Alexandria (to be distinguished from Theon of Smyrna) is dated by Suidas in the reign of Theodosius I (A.D. 379–395). His commentary on Ptolemy's *Syntaxis* is in eleven books, and his famous daughter Hypatia assisted in its revision. Pappus of Alexandria flourished in the reign of

circle, and with it the four right angles at the centre, are divided into 360 equal parts by radial lines. Each of these 360 *degrees* (μοῖραι or τμήματα) is divided into 60 equal parts called πρῶτα ἑξηκοστά, frequently represented as α′ ἑξηκοστά, *first sixtieths* or *minutes*. In turn each of these parts is divided into 60 δεύτερα ἑξηκοστά, or β′ ἑξηκοστά, *second sixtieths* or *seconds*. By further subdivision we obtain τρίτα ἑξηκοστά, or γ′ ἑξηκοστά, and so on. In similar manner the diameter of the circle is divided into 120 τμήματα, *segments*, each of these into sixtieths, and so on. The circular associations of the system tended to be forgotten, and it offered a convenient method for representing any number consisting of an integral number of units with fractional parts. The denominations of the parts might be written out in full (*e.g.*, πρῶτα ἑξηκοστὰ ϡ̄ = 900 minutes, α′ ἑξηκοστὰ σ̄ καὶ β′ ιε̄ = 200 minutes and 15 seconds), or a number consisting of degrees, minutes and seconds might be written down in three sets of numerals without any indication of the denominations other than is provided by the context (*e.g.*, ,αφιε̄ κ̄ ιε̄ = 1515° 20′ 15″).

After explaining the advantages of the notation owing to the large number of factors of 60, and noting the result of multiplying or dividing minutes by degrees, minutes by minutes, and so on, Theon gives an example of multiplication and then the two interesting passages which are now to be reproduced and translated :

Diocletian (A.D. 284–305). His chief work was his *Synagoge* or *Collection*, a handbook to Greek geometry which is now one of our main sources for the subject and will be extensively used in these pages.

(b) Division

Theon Alex. *in Ptol. Math. Syn. Comm.* i. 10, ed. Rome, *Studi e Testi*, lxxii. (1936), 461. 1–462. 17

Ἔστω δὲ καὶ ἀνάπαλιν δοθέντα ἀριθμὸν μερίσαι παρά τε μοίρας καὶ πρῶτα καὶ δεύτερα ἑξηκοστά. ἔστω ὁ δοθεὶς ἀριθμὸς ὁ ͵αφιε κ ιε· καὶ δέον ἔστω μερίσαι αὐτὸν παρὰ τὸν κε ιβ ι, τουτέστιν εὑρεῖν ποσάκις ἐστὶν ὁ κε ιβ ι ἐν τῷ ͵αφιε κ ιε.

Μερίζομεν αὐτὸν πρῶτον παρὰ τὸν ξ, ἐπειδήπερ ὁ παρὰ τὸν ξα ὑπερπίπτει καὶ ἀφαιροῦμεν ἑξηκοντάκι τόν τε κε καὶ τὸν ιβ, καὶ ἔτι τὸν ι. καὶ πρότερον τὸν κε, καὶ γίνονται ͵αφ· εἶτα ἐπὶ τῶν λοιπῶν μοιρῶν ιε κ ιε ἀναλύσαντες τὰς ιε μοίρας εἰς πρῶτα ἑξηκοστὰ καὶ προσθέντες αὐτοῖς τὰ πρῶτα ἑξηκοστὰ κ ἀπὸ τῶν γενομένων ϡκ πρῶτα πάλιν ἑξηκοστὰ ἀφαιροῦμεν ἑξηκοντάκις τὰ ιβ, τουτέστιν ψκ· καὶ ἔτι ἀπὸ τῶν λοιπῶν πρώτων ἑξηκοστῶν σ καὶ δευτέρων ιε ἀφαιροῦμεν ἑξηκοντάκις πάλιν τὰ ι· γίνεται δεύτερα μὲν ἑξηκοστὰ

---

[a] We may exhibit Theon's working as follows:

$$
\begin{array}{llll}
\text{1st division} & 25^\circ\ 12'\ 10'' \,\big)\ 1515^\circ & 20'\ 15'' & \big\lfloor\ 60^\circ \\
 & \quad 25^\circ.60^\circ = 1500^\circ & & \\
\hline
 & \qquad 15^\circ = 900' & & \\
 & \qquad\qquad 20' & & \\
\hline
 & \qquad\qquad 920' & & \\
 & \quad 12'.60^\circ = 720' & & \\
\hline
 & \qquad\qquad 200' & 15'' & \\
 & \quad 10''.60^\circ = 10' & &
\end{array}
$$

915

(*b*) Division

Theon of Alexandria, *Commentary on Ptolemy's Syntaxis*, i. 10, ed. Rome, *Studi e Testi*, lxxii. (1936), 461. 1–462. 17

Conversely, let it be required to divide a given number by a number expressed in degrees, minutes and seconds. Let the given number be 1515° 20′ 15″; and let it be required to divide this by 25° 12′ 10″, that is, to find how often 25° 12′ 10″ is contained in 1515° 20′ 15″.[a]

We take 60° as the first quotient, for 61° is too big; and we subtract sixty times 25° and sixty times 12′ and also sixty times 10″. Firstly, we take away sixty times 25°, which is 1500°. In the remainder, 15° 20′ 15″, we split up the 15° into minutes and add to them the 20′; and from the resulting 920′ we subtract sixty times 12′, that is, 720′. This leaves 200′ 15″, and we now subtract

2nd division 25° 12′ 10″ | 190′ 15″ | 7′

25°.7′ = 175′

15′ = 900″

15″

915″

12′.7′ = 84″

831″

10″.7′ = 1″ 10‴

3rd division 25° 12′ 10″ | 829″ 50‴ | 33″

25°.33″ = 825″

4″ 50‴ = 290‴

12′.33″ = 396‴

χ, πρῶτα δὲ ι. εἶτα πάλιν τὰ ὑπολιπέντα[1] πρῶτα ἑξηκοστὰ ρϛ καὶ δεύτερα ιε μερίζομεν παρὰ τὸν κε, καὶ γίνεται ὁ μερισμὸς παρὰ ζ· ὑπερπίπτε γὰρ παρὰ τὸν η. καὶ τὰ γενόμενα ἐκ τῆς παραβολῆς ἑξηκοστὰ πρῶτα ροε ἀφείλομεν ἀπὸ τῶν ρϛ πρώτων ἑξηκοστῶν. ἔπειτα τὰ λοιπὰ ιε πρῶτα ἑξηκοστὰ ἀναλύσαντες εἰς δεύτερα ϡ καὶ προσθέντες αὐτοῖς τὰ δεύτερα ἑξηκοστὰ ιε, ἀπὸ τῶν γενομένων ϡιε ἀφαιροῦμεν ἑπτάκις τὰ ιβ πρῶτα ἑξηκοστά, τουτέστιν πδ δεύτερα ἑξηκοστά, διὰ τὸ καὶ τὰ ζ πρῶτα εἶναι ἑξηκοστά. καὶ ὑπολείπεται λοιπὰ ωλα δεύτερα ἑξηκοστά. καὶ ἔτι ἀφελοῦμεν ὁμοίως ἑπτάκις καὶ τὰ ι δεύτερα ἑξηκοστά, ἃ γίνεται τρίτα ἑξηκοστὰ ο, τουτέστιν δεύτερον α καὶ τρίτα ι. καὶ λοιπὰ ὑπελίπη δεύτερα ἑξηκοστὰ ωκθ καὶ τρίτα ν. ταῦτα πάλιν παρὰ τὸν κε. καὶ γίνεται ὁ μὲν μερισμὸς παρὰ τὸν λγ, ἐκ δὲ τῆς παραβολῆς ωκε δεύτερα ἑξηκοστά. καὶ λοιπὰ ὑπελίπη δεύτερα ἑξηκοστὰ δ, τρίτα δὲ ν, ὁμοῦ δὲ τρίτα σϛ. ἔπειτα πάλιν ἀφείλομεν τὰ ιβ πρῶτα ἑξηκοστὰ τριακοντάκι καὶ τρὶς καὶ γίνεται τρίτα τϟϛ, ὡς ποιεῖν ἔγγιστα τὸν μερισμὸν τὸν ͵αφιε κ ιε παρὰ τὸν κε ιβ ι, ξ ζ λγ, ἐπεὶ καὶ ἐὰν ταῦτα πολλαπλασιάσωμεν ἐπὶ τὰ κε ιβ ι συνάγεται ὁ ͵αφιε κ ιε ἔγγιστα.

͵αφιε κ ιε   κε ιβ ι   ξ ζ λγ

(c) Extraction of Square Root

Ibid. 469. 16–473. 8

Τούτων θεωρηθέντων, ἑξῆς ἂν εἴη διαλαβεῖν πῶς

sixty times 10″; that is 600″, or 10′. The remainder is 190′ 15″, and, making a new start, we divide by 25°; the quotient is 7′, for 8′ is too big. The number resulting from this division is 175′, which we subtract from the 190′. There is a remainder of 15′, which we split up into 900″ and to it add the 15″; from the resulting 915″ we subtract seven times 12′, which is 84″ on account of the seven being minutes; there is left a remainder 831″. Similarly we subtract seven times 10″, which is 70‴, or 1″ 10‴. The remainder is 829″ 50‴. We divide this in turn by 25°. The quotient is 33″, and the number resulting from the division is 825″, leaving a remainder of 4″ 50‴, or 290‴. Next we subtract thirty-three times 12′, which is 396‴. Thus the quotient obtained by dividing 1515° 20′ 15″ by 25° 12′ 10″ is approximately 60° 7′ 33″, inasmuch as, if we multiply this quotient by 25° 12′ 10″, the result will be approximately 1515° 20′ 15″.

1515° 20′ 15″	25° 12′ 10″	60° 7′ 33″

(c) Extraction of Square Root

Ibid. 469. 16–473. 8

After this demonstration the next step is to inquire

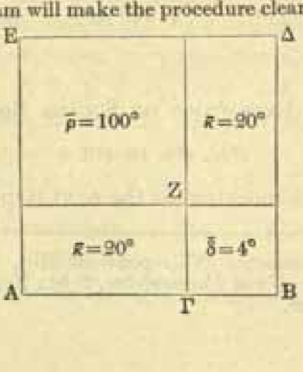
---

[1] "Forme suspecte. Voir pourtant Hirt, *Handbuch der griechischen Laut- und Formenlehre*, 2e éd., Heidelberg, 1912, p. 506."—Rome.

ἂν δοθέντος χωρίου τινὸς τετραγώνου μὴ ἔχοντος πλευρὰν μήκει ῥητὴν τὴν σύνεγγυς αὐτοῦ τετραγωνικὴν πλευρὰν ἐπιλογισώμεθα. καί ἐστιν τὸ τοιοῦτον δῆλον ἐπὶ ῥητὴν ἔχοντος πλευράν, ἐκ τοῦ δ′ θεωρήματος τοῦ β′ βιβλίου τῶν Στοιχείων, οὗ ἡ πρότασίς ἐστιν τοιαύτη· ἐὰν εὐθεῖα γραμμὴ τμηθῇ ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης τετράγωνον ἴσον ἐστὶν τοῖς τε ἀπὸ τῶν τμημάτων τετραγώνοις καὶ τῷ δὶς ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογωνίῳ. ἐὰν γὰρ ἔχοντες δοθέντα ἀριθμὸν τετράγωνον ὡς τὸν ρ̅μ̅δ̅, ῥητὴν ἔχοντα πλευρὰν ὡς τὴν ΑΒ εὐθεῖαν, καὶ λαβόντες αὐτοῦ ἐλάσσονα τετράγωνον τὸν ρ̅, οὗ ἐστιν πλευρὰ ι̅, καὶ ὑποθέμενοι τὴν ΑΓ ι̅, διπλασιάσαντες αὐτὴν [καὶ][1] διὰ τὸ δὶς ὑπὸ τῶν ΑΓ, ΓΒ, ⟨παρὰ⟩[2] τὰ γενόμενα κ̅ παραβάλωμεν [παρὰ][3] τὰ λοιπὰ μ̅δ̅, τῶν ὑπολειπομένων δ̅ ἔσται τὸ ἀπὸ τῆς ΓΒ, αὕτη δὲ μήκει β̅· ἦν δὲ καὶ ἡ ΑΓ ι̅· καὶ ὅλη ἄρα ἡ ΑΒ ἔσται μοιρῶν ι̅β̅, ὅπερ ἔδει δεῖξαι.

[1] καὶ om. Rome. [2] παρὰ add. Rome.
[3] παρὰ om. Rome.

[a] The diagram will make the procedure clear. The square



in what manner, given the area of a square whose side is irrational, we may make an approximation to its side. In the case of a square with a rational side the method is clear from the fourth theorem of the second book of the *Elements*, whose enunciation is as follows: *If a straight line be cut at random, the square on the whole is equal to the squares on the segments, and twice the rectangle contained by the segments.* For if the given number is a square such as 144, having a rational side AB, we take the square 100, which is less than 144 and has 10 as its side, and make AΓ equal to 10. Doubling it, because the rectangle contained by AΓ, ΓB is taken twice, we get 20, and by this number we divide the remainder 44, obtaining a remainder 4 as the square on ΓB, whose length will therefore be 2. Now AΓ was 10, and therefore the whole AB is 12, which was to be proved.[a]

AΔ is divided up into the squares EZ, BZ and the equal rectangles AZ, ZΔ.

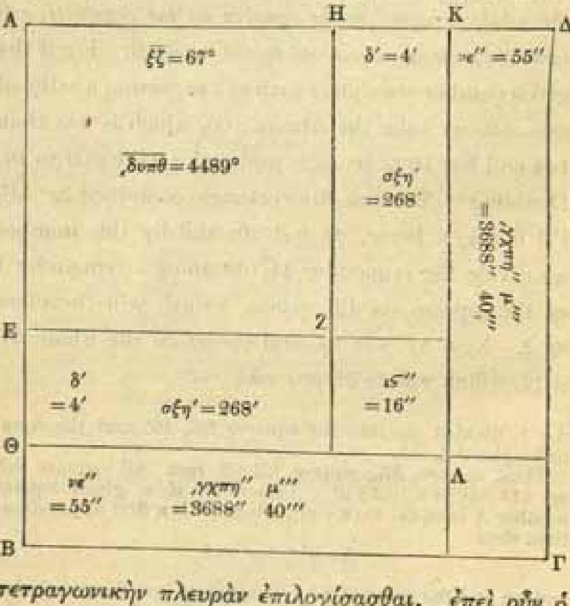
Thus, square AΔ = square EZ + 2 rect. AZ + square BZ or $144 = 10^2 + 2.10.2 + 2^2$. Generally, if a given square number A is equal to $(a+x)^2$, where a^2 is a first approximation, then

$$A = a^2 + 2ax + x^2$$

and we find the value of x by dividing $2a$ into the remainder when a^2 is subtracted from A.

If A is not a square number, then this gives a method of finding an approximation, $a + x$, to the square root.

Ἵνα δὲ καὶ ἐπί τινος τῶν ἐν τῇ Συντάξει παρακειμένων ἀριθμῶν ὑπ' ὄψιν ἡμῖν γένηται ἡ τῆς κατὰ μέρος ἀφαιρέσεως διάκρισις, ποιησόμεθα τὴν ἀπόδειξιν ἐπὶ τοῦ ͵δφ ἀριθμοῦ, οὗ τὴν πλευρὰν ἐξέθετο μοιρῶν ξζ δ νε. ἐκκείσθω χωρίον τετράγωνον τὸ ΑΒΓΔ, δυνάμει μόνον ῥητόν, οὗ τὸ ἐμβαδὸν ἔστω μοιρῶν ͵δφ, καὶ δέον ἔστω τὴν σύνεγγυς αὐτοῦ



τετραγωνικὴν πλευρὰν ἐπιλογίσασθαι. ἐπεὶ οὖν ὁ

[a] The method which Theon proposes to use may be summarized as follows. A first approximation to the square root

In order to show visually, for one of the numbers in the *Syntaxis*, this extraction of the root by taking away the parts, we shall construct the proof for the number 4500°, whose side he [Ptolemy] made 67° 4′ 55″. Let ΑΒΓΔ be a square area, the square alone being rational, and let its contents be 4500°, and let it be required to calculate the side of a square approximating to it.[a] Since the square

[a] of 4500 is 67, for $67^2=4489$. (This suggests that Theon may have had a table of squares before him.) Theon proposes to find the square root of 4500 in the form $67+\frac{x}{60}+\frac{y}{60^2}$. That is,

$$\sqrt{4500}=\sqrt{67^2+11}=67+\frac{x}{60}+\frac{y}{60^2}.$$

It follows from Euclid ii. 4 that $\frac{2.67x}{60}$ must be less than 11, or x must be less than $\frac{660}{2.67}$. The nearest whole number obtained by dividing 2.67 into 660 is 4, and we try 4 for the value of x. On trial it is found that 4 satisfies the conditions of the problem, for $\left(67+\frac{4}{60}\right)^2$ is less than 4500, the remainder being $\frac{7424}{60^2}$. Theon proves this geometrically. If $\text{AE}=67$, then the square $\text{AZ}=4489$ and the gnomon ΒΖΖΔ is therefore 11, or $\frac{660}{60}$. Putting $\text{E}\Theta=\text{HK}=\frac{4}{60}$, we have rect. $\Theta\text{Z}=$ rect. $\text{ZK}=\frac{4.67}{60}=\frac{268}{60}$. Their sum is $\frac{536}{60}$ and this we subtract from $\frac{660}{60}$, getting $\frac{124}{60}$ or $\frac{7440}{60^2}$. From this we subtract $\frac{16}{60^2}$, being the value of the square ΖΛ, and so get $\frac{7424}{60^2}$ for the remaining gnomon ΒΛΛΔ, as was stated above. This remainder now serves as a basis to obtain the third term y of the quotient. Since $\left\{\left(67+\frac{4}{60}\right)+\frac{y}{60^2}\right\}^2$ is approximately 4500, we have by

σύνεγγυς τοῦ ͵δφ τετράγωνος ῥητὴν ἔχων πλευρὰν ὅλων μονάδων ἐστὶν ͵δυπθ ἀπὸ πλευρᾶς τοῦ ξζ, ἀφῃρήσθω ἀπὸ τοῦ ΑΒΓΔ τετραγώνου τὸ ΑΖ τετράγωνον μονάδων ͵δυπθ, οὗ ἡ πλευρὰ ἔστω μονάδων ξζ· ὁ λοιπὸς ἄρα ὁ ΒΖΖΔ γνώμων ἔσται μονάδων ια, ἃς ἀναλύσαντες εἰς πρῶτα ἑξηκοστὰ χξ ἐκθησόμεθα. ἔπειτα διπλασιάσαντες τὴν ΕΖ διὰ τὸ δὶς ὑπὸ ΕΖ, ὥσπερ ἐπ' εὐθείας τῆς ΕΖ τὴν ΖΗ λαμβάνοντες, παρὰ τὰ γενόμενα ρλδ παραβαλοῦμεν τὰ χξ ἑξηκοστὰ πρῶτα, καὶ τῶν γενομένων ἐκ τῆς παραβολῆς δ πρώτων ἑξηκοστῶν ἕξομεν ἑκατέραν τῶν ΕΘ, ΗΚ. καὶ ἀναπληρώσαντες τὰ ΘΖ, ΖΚ παραλληλόγραμμα ἕξομεν καὶ αὐτὰ φλϛ πρώτων ἑξηκοστῶν, ἑκάτερον δὲ ὂν σξη. εἶτα πάλιν τὰ ὑπολιπέντα ρκδ πρῶτα ἑξηκοστὰ ἀναλύσαντες εἰς δεύτερα ͵ζυμ, ἀφελοῦμεν καὶ τὸ ΖΛ ἀπὸ πρώτων δ γενόμενον ἑξηκοστῶν δευτέρων ιϛ, ἵνα γνώμονα περιθέντες τῷ ἐξ ἀρχῆς τετραγώνῳ τῷ ΑΖ ἔχωμεν τὸ ΑΛ τετράγωνον ἀπὸ πλευρᾶς ξζ δ συναγόμενον μοιρῶν ͵δυϟζ νϛ ιϛ. καὶ λοιπὸν πάλιν τὸν ΒΛΛΔ γνώμονα μοιρῶν β γ μδ, τουτέστιν δευτέρων ἑξηκοστῶν ͵ζυκδ. ἔτι δὲ πάλιν διπλασιάσαντες τὴν ΘΛ ὡς ἐπ' εὐθείας τυγχανούσης τῇ ΘΛ τῆς ΛΚ, καὶ παρὰ τὰ γινόμενα ρλδ η μερίσαντες τὰ ͵ζυκδ δεύτερα ἑξηκοστά, τῶν ἐκ τῆς παραβολῆς γενομένων νε ἔγγιστα δευτέρων ἑξηκοστῶν ἔχομεν

Euclid ii. 4 that $2\left(67+\frac{4}{60}\right)\cdot\frac{y}{60^2}+\left(\frac{y}{60^2}\right)^2$ is approximately $\frac{7424}{60^2}$, and we obtain a trial value for y by dividing $2\left(67+\frac{4}{60}\right)$ or

which approximates to 4500° but has a rational side and consists of a whole number of units is 4489° on a side of 67°, let the square AZ, with area 4489° and side 67°, be taken away from the square ΑΒΓΔ. The remainder, the gnomon ΒΖΖΔ, will therefore be 11°, which we reduce to 660′ and set out. Then we double EZ, because the rectangle on EZ has to be taken twice, as though we regarded ZH as on the straight line EZ, divide the result 134° into 660′, and by the division get 4′, which gives us each of EΘ, HK. Completing the parallelograms ΘZ, ZK, we have for their sum 536′, or 268′ each. Continuing, we reduce the remainder, 124′, into 7440″, and subtract from it also the complement ZΛ, which is 16″, in order that by adding a gnomon to the original square AZ we may have the square AΛ on a side 67° 4′ and consisting of 4497° 56′ 16″. The remainder, the gnomon ΒΛΛΔ, consists of 2° 3′ 44″, that is, 7424″. Continuing the process, we double ΘΛ, as though ΛK were in a straight line with ΘΛ and equal to it, divide the product 134° 8′ into 7424″, and the result is approximately 55″, which gives

---

$\left(134+\frac{8}{60}\right)$ into 7424, which yields $y=55$. Putting $\frac{55}{60^2}$ as the value of ΘB, KΔ, we get the value $\frac{3688}{60^2}+\frac{40}{60^3}$ for each of the rects. BΛ, ΛΔ, or $\frac{7377}{60^2}+\frac{20}{60^3}$ for their sum. Subtracting this from $\frac{7424}{60^2}$ we get $\frac{46}{60^2}+\frac{40}{60^3}$, which Theon notes will be approximately the value of the square ΛΓ, or $\left(\frac{55}{60^2}\right)^2$. As a matter of fact, $\frac{46}{60^2}+\frac{40}{60^3}=\frac{2800}{60^3}=\frac{16800}{60^4}$ while $\left(\frac{55}{60^2}\right)^2=\frac{3025}{60^4}$.

ἔγγιστα ἑκατέραν τῶν ΘΒ, ΚΔ. καὶ συμπληρώσαντες τὰ ΒΛ, ΛΔ παραλληλόγραμμα, ἕξομεν καὶ αὐτὰ ἑξηκοστῶν δευτέρων μὲν ͵ζτο καὶ τρίτων υμ̄, ἑκάτερον δὲ δευτέρων μὲν ἑξηκοστῶν ͵γχπε καὶ τρίτων σκ̄.[1] καὶ λοιπὰ ὑπελίπη ἑξηκοστὰ δεύτερα μϛ̄ καὶ τρίτα μ̄, ἅπερ ἔγγιστα ποιεῖ τὸ ΛΓ τετράγωνον, ἀπὸ πλευρᾶς τυγχάνον ϛ̄ε̄ δευτέρων ἑξηκοστῶν, καὶ ἔσχομεν τὴν πλευρὰν τοῦ ΑΒΓΔ τετραγώνου, μοιρῶν τυγχάνοντος ͵δφ, ξ̄ζ̄ δ̄ νε ἔγγιστα.

Ὥστε καὶ καθόλου ἐὰν ζητῶμεν ἀριθμοῦ τινος τὴν τετραγωνικὴν πλευρὰν ἐπιλογίσασθαι, λαμβάνομεν πρῶτον τοῦ σύνεγγυς τετραγώνου ἀριθμοῦ τὴν πλευράν. εἶτα ταύτην διπλασιάσαντες καὶ παρὰ τὸν γινόμενον ἀριθμὸν μερίσαντες τὸν λοιπὸν ἀριθμὸν ἀναλυθέντα εἰς πρῶτα ἑξηκοστά, καὶ ἀπὸ τοῦ ἐκ τῆς παραβολῆς γενομένου ἀφελοῦμεν τετράγωνον, καὶ ἀναλύοντες πάλιν τὰ ὑπολειπόμενα εἰς δεύτερα ἑξηκοστά, καὶ μερίζοντες παρὰ τὸν διπλασίονα τῶν μοιρῶν καὶ ἑξηκοστῶν, ἕξομεν ἔγγιστα τὸν ἐπιζητούμενον τῆς πλευρᾶς τοῦ τετραγώνου χωρίου ἀριθμόν.

(d) Extraction of Cube Root

Heron, *Metr.* iii. 20, ed. Schöne 178. 3-16

Ὡς δὲ δεῖ λαβεῖν τῶν ρ̄ μονάδων κυβικὴν πλευρὰν νῦν ἐροῦμεν.

[1] So the oldest ms. In others the numbers are worked out to the equivalent forms ͵ζτοζ″ κ‴. ͵γχπη″ μ‴.

[a] In the Greek of the oldest ms. the numbers are given as 7370″ 440‴ and 3685″ 220‴, in which form Theon would first

us an approximation to ΘB, KΔ. Completing the parallelograms BΛ, ΛΔ, we shall have for their joint area 7377″ 20‴, or 3688″ 40‴ each.[a] The remainder is 46″ 40‴, which approximates to the square ΛΓ on a side of 55″, and so we obtain for the side of the square ΑΒΓΔ, consisting of 4500°, the approximation 67° 4′ 55″.

In general, if we seek the square root of any number, we take first the side of the nearest square number, double it, divide the product into the remainder reduced to minutes, and subtract the square of the quotient; proceeding in this way we reduce the remainder to seconds, divide it by twice the quotient in degrees and minutes, and we shall have the required approximation to the side of the square area.[b]

(d) Extraction of Cube Root

Heron, *Metrics* iii. 20, ed. Schöne 178. 3-16

We shall now inquire into the method of extracting the cube root of 100.

obtain them. In other MSS. the numbers are worked out to the form 7377″ 20‴, 3688″ 40‴.

[b] In his Table of Chords Ptolemy gives the approximation

$$\sqrt{3}=\frac{103}{60}+\frac{55}{60^2}+\frac{23}{60^3},$$

which is equivalent to 1·7320509 and is correct to six decimal places. This formula could be obtained by a slight adaptation of Theon's method.

Archimedes gives, without any explanation, the following approximation:

$$\frac{1351}{780}>\sqrt{3}>\frac{265}{153}.$$

The formula opens up a wide field of conjecture. See Heath, *The Works of Archimedes*, pp. lxxx-xcix.

Λαβὲ τὸν ἔγγιστα κύβον τοῦ $\overline{\rho}$ τόν τε ὑπερβάλλοντα καὶ τὸν ἐλλείποντα· ἔστι δὲ ὁ $\overline{\rho\kappa\epsilon}$ καὶ ὁ $\overline{\xi\delta}$. καὶ ὅσα μὲν ὑπερβάλλει, μονάδες $\overline{\kappa\epsilon}$, ὅσα δὲ ἐλλείπει, μονάδες $\overline{\lambda\varsigma}$. καὶ ποίησον τὰ $\overline{\epsilon}$ ἐπὶ τὰ $\overline{\lambda\varsigma}$· γίγνεται $\overline{\rho\pi}$· καὶ τὰ $\overline{\rho}$· γίγνεται $\overline{\sigma\pi}$. ⟨καὶ παράβαλε τὰ $\overline{\rho\pi}$ παρὰ τὰ $\overline{\sigma\pi}$.⟩[1] γίγνεται $\overset{\iota\delta'}{\theta}$. πρόσβαλε τῇ [κατὰ] τοῦ ἐλάσσονος κύβου πλευρᾷ, τουτέστι τῷ $\overline{\delta}$· γίγνεται μονάδες $\overline{\delta}$ καὶ $\overset{\iota\delta'}{\theta}$ τοσούτων ἔσται ἡ τῶν $\overline{\rho}$ μονάδων κυβικὴ πλευρὰ ὡς ἔγγιστα.

[1] καὶ παράβαλε τὰ $\overline{\rho\pi}$ παρὰ τὰ $\overline{\sigma\pi}$ supplevit H. Schöne.

---

[a] If p^3 and q^3 are the two cube numbers between which A lies, and $A = p^3 - a = q^3 + b$, then Heron's formula can be generalized as follows:

$$\sqrt[3]{A} = q + \frac{b\sqrt{a}}{A + b\sqrt{a}}.$$

It is unlikely that Heron worked with this general formula; his method was probably empirical. The subject is discussed

Take the nearest cube in excess of 100 and also the nearest which is deficient; they are 125 and 64. The excess of the former is 25, the deficiency of the latter 36. Now multiply 36 by 5; the result is 180; and adding 100 gives 280. Dividing 180 by 280 gives $\frac{9}{14}$. Add this to the side of the lesser cube, that is, to 4, and the result is $4\frac{9}{14}$. This [a] is the closest approximation to the cube root of 100.

by M. Curtze, *Quadrat-und Kubikwurzeln bei den Griechen nach Herons neu aufgefundenen* Μετρικά (*Zeitschrift f. Math. u. Phys.* xlii., 1897, *Hist.-lit. Abth.*, pp. 113-120), G. Wertheim, *Herons Ausziehung der irrationalen Kubikwurzeln* (*ibid.* xliv., 1899, *Hist.-lit. Abth.*[3], pp. 1-3), and G. Eneström, *Bibliotheca Mathematica*, viii., 1907–1908, pp. 412-413. The actual value of $(4\frac{9}{14})^3$ is $100\frac{185}{2744}$.

There is no example in Greek mathematics of the extraction of a cube root fully worked out by means of the formula $(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$, corresponding to Theon's method for square roots; but by means of this formula Philon of Byzantium (*Mech. Synt.* iv. 6-7, ed. R. Schöne) appears to have approximated to the cube roots of 1500, 2000, 3000, 5000 and 6000. Heron (*Metrica* iii. 22, ed. H. Schöne 184. 1-2) gives without explanation 46 as the cube root of 97050.

III. PYTHAGOREAN ARITHMETIC

III. PYTHAGOREAN ARITHMETIC

(a) First Principles

Eucl. *Elem.* vii.

Ὅροι

α΄. Μονάς ἐστιν, καθ' ἣν ἕκαστον τῶν ὄντων ἓν λέγεται.

β΄. Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.

γ΄. Μέρος ἐστὶν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρῇ τὸν μείζονα.

δ΄. Μέρη δέ, ὅταν μὴ καταμετρῇ.

ε΄. Πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάσσονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος.

ϛ΄. Ἄρτιος ἀριθμός ἐστιν ὁ δίχα διαιρούμενος.

ζ΄. Περισσὸς δὲ ὁ μὴ διαιρούμενος δίχα ἢ [ὁ] μονάδι διαφέρων ἀρτίου ἀριθμοῦ.

η΄. Ἀρτιάκις ἄρτιος ἀριθμός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν.

---

[a] The theory of numbers is treated by Euclid in Books vii.-x. The definitions prefixed to Book vii. are wholly Pythagorean in their outlook, though there are differences in

III. PYTHAGOREAN ARITHMETIC

(a) First Principles

Euclid, *Elements* vii.

DEFINITIONS [a]

1. A *unit* is that in virtue of which each of the things that exist is called one.
2. A *number* is a multitude composed of units.
3. A number is a *part* of a number, the less of the greater, when it measures the greater.
4. But *parts*, when it does not measure it.
5. The greater number is a *multiple* of the less when it is measured by the less.
6. An *even number* is one that is divisible into two equal parts.
7. An *odd number* is one that is not divisible into two equal parts, or that differs from an even number by a unit.
8. An *even-times even number* [b] is one that is measured by an even number according to an even number.

detail. Heath's notes (*The Thirteen Books of Euclid's Elements*, vol. ii. pp. 279-295) are invaluable.

[b] It is a consequence of this definition that an even-times even number may also be even-times odd, as 24 is both 6×4 and 8×3 (*cf.* Euclid ix. 34, where it is proved that this must be so for certain numbers). Three later writers, Nicomachus, Theon of Smyrna and Iamblichus, defined an even-times even number differently, as a number of the form 2^p.

θʹ. Ἀρτιάκις δὲ περισσός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

[ιʹ. Περισσάκις ἄρτιός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν.][1]

ιαʹ. Περισσάκις δὲ περισσὸς ἀριθμός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

ιβʹ. Πρῶτος ἀριθμός ἐστιν ὁ μονάδι μόνῃ μετρούμενος.

ιγʹ. Πρῶτοι πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ μονάδι μόνῃ μετρούμενοι κοινῷ μέτρῳ.

ιδʹ. Σύνθετος ἀριθμός ἐστιν ὁ ἀριθμῷ τινι μετρούμενος.

ιεʹ. Σύνθετοι δὲ πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ ἀριθμῷ τινι μετρούμενοι κοινῷ μέτρῳ.

[1] ιʹ. περισσάκις . . . ἀριθμόν om. Heiberg.

---

[a] Instead of Euclid's term ἀρτιάκις περισσός, Nicomachus, Theon and Iamblichus used the single word ἀρτιοπέριττος. According to Nicomachus (*Arith. Introd.* i. 9) such a number, when divided by 2, leaves an odd number as the quotient, *i.e.*, it is of the form $2(2n+1)$. In this later subdivision an *odd-even* (περισσάρτιος) number is one which can be halved twice or more successively, but the final quotient is always an odd number and not unity, *i.e.*, a number of the form $2^{p+1}(2n+1)$. We thus have three mutually exclusive classes of even numbers: (1) *even-even*, of the form 2^p; (2) *even-odd*, of the form $2(2n+1)$; and (3) *odd-even*, of the form $2^{p+1}(2n+1)$, where (1) and (3) are extremes and (2) partakes of the nature of both. The *odd-odd* is not defined by Nicomachus and Iamblichus, but according to a curious usage in Theon it is one of the names applied to prime numbers, for these have two odd factors, 1 and the number itself.

[b] According to this definition, any even-times odd number would also be odd-times even. The definition appears to have been known to Iamblichus, but there can be little doubt

9. An *even-times-odd number*[a] is one that is measured by an even number according to an odd number.

[10. An *odd-times even number* is one that is measured by an odd number according to an even number.][b]

11. An *odd-times odd number* is one that is measured by an odd number according to an odd number.

12. A *prime number* is one that is measured by the unit alone.

13. Numbers *prime to one another* are those which are measured by a unit alone as a common measure.

14. A *composite number* is one that is measured by some number.

15. Numbers *composite to one another* are those which are measured by some number as a common measure.[c]

that it is an interpolation. If both definitions are genuine, one is not only pointless but the enunciations of ix. 33 and ix. 34 become difficult to understand, and were, indeed, read differently by Iamblichus from what we find in our MSS. We have to choose between accepting Iamblichus's reading in all three places and rejecting Def. 10 as interpolated. I agree with Heiberg (*Euklid-Studien*, pp. 198 *et seq.*) that the definition was probably interpolated by someone who was unaware of the difference between the Euclidean and the later Pythagorean classifications, but noticed the absence of a definition by Euclid of an odd-times even number and tried to supply one.

[c] Euclid's definition of prime and composite numbers differs greatly from the classification of Nicomachus (*Arith. Introd.* i. 11-13) and Iamblichus. To match the three classes of even numbers, they devised three classes of odd numbers: (1) *πρῶτον καὶ ἀσύνθετον*, *prime and incomposite*, which is a prime number in the Euclidean sense; (2) *δεύτερον καὶ σύνθετον*, *secondary and composite*, which appears to be the product of prime numbers; and (3) *ὃ καθ' ἑαυτὸ μὲν δεύτερον καὶ σύνθετον, πρὸς ἄλλο δὲ πρῶτον καὶ ἀσύνθετον*, *that which is secondary and composite in itself, but prime and incomposite in relation to another*, where all the factors must

ιϛ΄. Ἀριθμὸς ἀριθμὸν πολλαπλασιάζειν λέγεται, ὅταν, ὅσαι εἰσὶν ἐν αὐτῷ μονάδες, τοσαυτάκις συντεθῇ ὁ πολλαπλασιαζόμενος, καὶ γένηταί τις.

ιζ΄. Ὅταν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος ἐπίπεδος καλεῖται, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

ιη΄. Ὅταν δὲ τρεῖς ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος στερεός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

ιθ΄. Τετράγωνος ἀριθμός ἐστιν ὁ ἰσάκις ἴσος ἢ [ὁ] ὑπὸ δύο ἴσων ἀριθμῶν περιεχόμενος.

κ΄. Κύβος δὲ ὁ ἰσάκις ἴσος ἰσάκις ἢ [ὁ] ὑπὸ τριῶν ἴσων ἀριθμῶν περιεχόμενος.

κα΄. Ἀριθμοὶ ἀνάλογόν εἰσιν, ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάκις ἢ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ὦσιν.

κβ΄. Ὅμοιοι ἐπίπεδοι καὶ στερεοὶ ἀριθμοί εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς.

κγ΄. Τέλειος ἀριθμός ἐστιν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὤν.

---

be odd and prime. The classification is defective, as (2) includes (3). Another defect is that the term *composite* is restricted to odd numbers instead of being given, as by Euclid, its general signification. For an earlier and different use of the terms by Speusippus, see *infra*, p. 78 n. *a*.

[a] For figured numbers, see *infra*, pp. 86-99.

[b] "Ἀνάλογον, though usually written in one word, is equivalent to ἀνὰ λόγον, *in proportion*. It comes, however, in

16. A number is said to *multiply* a number when that which is multiplied is added to itself as many times as there are units in the other, and so some number is produced.

17. And when two numbers have multiplied each other so as to make some number, the resulting number is called *plane*, and its sides are the numbers which have multiplied each other.[a]

18. And when three numbers have multiplied each other so as to make some number, the resulting number is *solid*, and its sides are the numbers which have multiplied each other.

19. A *square number* is equal multiplied by equal, or one that is contained by two equal numbers.

20. And a *cube* is equal multiplied by equal and again by equal, or a number that is contained by three equal numbers.

21. Numbers are *proportional* [b] when the first is the same multiple, or the same part, or the same parts, of the second as the third is of the fourth.

22. *Similar plane* and *solid* numbers are those which have their sides proportional.

23. A *perfect number* [c] is one that is equal to [the sum of] its own parts.

Greek mathematics to be used practically as an indeclinable adjective. . . . Sometimes it is used adverbially " (Heath, *The Thirteen Books of Euclid's Elements*, vol. ii. p. 129).

[b] This definition, inasmuch as it depends on the notion of a part of a number, is applicable only to commensurable magnitudes. A new definition, applicable to incommensurable as well as commensurable magnitudes, and due in substance though not necessarily in form to Eudoxus, is given by Euclid in *Elements* v. Def. 5 (see *infra*, pp. 444-447).

[c] The term " perfect number " was apparently not used in this sense before Euclid. The subject is treated *infra*, pp. 74-87.

(b) Classification of Numbers

Philolaus ap. Stob. *Ecl.* i. 21. 7c, ed. Wachsmuth 188. 9-12; Diels, *Vors.* i^5. 408. 7-10

Ἐκ τοῦ Φιλολάου Περὶ κόσμου . . .

"Ὅ γα μὰν ἀριθμὸς ἔχει δύο μὲν ἴδια εἴδη, περισσὸν καὶ ἄρτιον, τρίτον δὲ ἀπ᾽ ἀμφοτέρων μειχθέντων ἀρτιοπέριττον· ἑκατέρω δὲ τῶ εἴδεος πολλαὶ μορφαί, ἃς ἕκαστον αὐταυτὸ σημαίνει."

Nicom. *Arith. Introd.* i. 7, ed. Hoche 13. 7–14. 12

Ἀριθμός ἐστι πλῆθος ὡρισμένον ἢ μονάδων σύστημα ἢ ποσότητος χύμα ἐκ μονάδων συγκείμενον, τοῦ δὲ ἀριθμοῦ πρώτη τομὴ τὸ μὲν ἄρτιον, τὸ δὲ περιττόν. ἔστι δὲ ἄρτιον μέν, ὃ οἷόν τε εἰς δύο ἴσα διαιρεθῆναι μονάδος μέσον μὴ παρεμπιπτούσης, περιττὸν δὲ τὸ μὴ δυνάμενον εἰς δύο ἴσα μερισθῆναι διὰ τὴν προειρημένην τῆς μονάδος μεσιτείαν. οὗτος μὲν οὖν ὁ ὅρος ἐκ τῆς δημώδους ὑπολήψεως· κατὰ δὲ τὸ Πυθαγορικὸν ἄρτιος ἀριθμός ἐστιν ὁ τὴν εἰς τὰ μέγιστα καὶ τὰ ἐλάχιστα κατὰ ταὐτὸ τομὴν ἐπιδεχόμενος, μέγιστα μὲν πηλικότητι, ἐλάχιστα δὲ ποσότητι, κατὰ φυσικὴν τῶν δυὸ τούτων γενῶν ἀντιπεπόνθησιν, περισσὸς δὲ ὁ μὴ δυνάμενος τοῦτο παθεῖν, ἀλλ᾽ εἰς ἄνισα δύο τεμνόμενος. ἑτέρῳ δὲ τρόπῳ κατὰ τὸ παλαιὸν

---

[a] The "even-odd" would seem to mean here the product of odd and even numbers. This agrees with Euclid's usage in *Elem.* vii. Def. 9. For the later specialized Pythagorean meaning, see *supra*, p. 68 n. *a*.

[b] If an odd number is set out as $2n+1$ units in a straight line, then it can be divided into two sections of n units

(b) Classification of Numbers

Philolaus, cited by Stobaeus, *Extracts* i. 21. 7c, ed. Wachsmuth 188. 9-12; Diels, *Vors.* i[3]. 408. 7-10

From Philolaus's book *On the Universe* . . .

"Number is of two special kinds, odd and even, with a third, even-odd,[a] arising from a mixture of both; and of each kind there are many forms, which each thing exhibits in itself."

Nicomachus, *Introduction to Arithmetic* i. 7, ed. Hoche 13. 7–14. 12

Number is a determinate multitude or collection of units or flow of quantity made up of units, and the first division of number is into the even and odd. Now the even is that which can be divided into two equal parts, without a unit inserting itself in the middle, while the odd is that which cannot be divided into two equal parts owing to the unit inserting itself as aforesaid.[b] This is the definition commonly accepted; but according to the Pythagoreans an even number is that which is divided, by one and the same operation, into the greatest and the least parts, greatest in size but least in quantity,[c] in accordance with a natural reciprocity of the two species, while an odd number cannot be so divided but is only divisible into two unequal parts. There is another ancient way of defining an even number

measured from either end, with a single unit left over in the middle; but an even number of $2n$ units can be divided into two equal sections with no unit left over in the middle.

[c] *i.e.* into two halves, for there cannot be any part greater than half nor fewer parts than two.

ἄρτιός ἐστιν ὁ καὶ εἰς δύο ἴσα τμηθῆναι δυνάμενος καὶ εἰς ἄνισα δύο, πλὴν τῆς ἐν αὐτῷ ἀρχοειδοῦς δυάδος θάτερον τὸ διχοτόμημα μόνον ἐπιδεχομένης τὸ εἰς ἴσα, ἐν ἥτινι οὖν τομῇ παρεμφαίνων τὸ ἕτερον εἶδος μόνον τοῦ ἀριθμοῦ, ὅπως ἂν διχασθῇ, ἀμέτοχον τοῦ λοιποῦ· περισσὸς δέ ἐστιν ἀριθμὸς ὁ καθ' ἡντιναοῦν τομὴν εἰς ἄνισα πάντως γινομένην ἀμφότερα ἅμα ἐμφαίνων τὰ τοῦ ἀριθμοῦ δυὸ εἴδη οὐδέποτε ἄκρατα ἀλλήλων, ἀλλὰ πάντοτε σὺν ἀλλήλοις. ἐν δὲ τῷ δι' ἀλλήλων ὅρῳ περιττός ἐστιν ὁ μονάδι ἐφ' ἑκάτερα διαφέρων ἀρτίου ἀριθμοῦ, τουτέστιν ἐπὶ τὸ μεῖζον καὶ ἔλαττον, ἄρτιος δὲ ὁ μονάδι διαφέρων ἐφ' ἑκάτερον περισσοῦ ἀριθμοῦ, τουτέστι μονάδι μείζων καὶ μονάδι ἐλάσσων.

(c) Perfect Numbers

[Iambl.] *Theol. Arith.*, ed. de Falco 82. 10–85. 23; Diels, *Vors.* i[5]. 400. 22–402. 11

Ὅτι καὶ Σπεύσιππος, ὁ Πωτώνης μὲν υἱὸς τῆς τοῦ Πλάτωνος ἀδελφῆς, διάδοχος δὲ Ἀκαδημείας πρὸ Ξενοκράτου, ἐκ τῶν ἐξαιρέτως σπουδασθεισῶν ἀεὶ Πυθαγορικῶν ἀκροάσεων, μάλιστα δὲ τῶν

---

[a] It is probable that we have here a trace of an original conception according to which 2 (the dyad) was regarded as being, not a number, but the principle or beginning of the even, just as 1 was not regarded as a number, but the principle or beginning of number; for the qualification about the dyad seems clearly to be a later addition to the original definition. It must, however, have been pre-Platonic, for in *Parm.* 143 D Plato speaks of 2 as even. Aristotle, who adds (*Topics* Θ 2, 157 a 39) that 2 is the only even number which is prime, says (*Met.* A 5, 986 a 19) the Pythagoreans regarded the One as

according to which it can be divided both into two equal parts and into two unequal parts, save in the case of the fundamental dyad, which can be divided only into two equal parts [a]; but howsoever it be divided, it must have its two parts of the same kind,[b] without partaking of the other kind; while the odd is that which, howsoever it be divided, always yields two unequal parts and so exhibits at one and the same time both species of number, never independent of one another but always together.[c] To give a definition in terms one of another, the odd is that which differs from even number by a unit in both directions, that is, in the direction both of the greater and of the lesser, while the even is that which differs by a unit from odd number in either direction, that is, it is greater by a unit and less by a unit.

(c) Perfect Numbers

[Iamblichus], *Theologumena Arithmeticae*, ed. de Falco 82. 10–85. 23; Diels, *Vors.* i[5]. 400. 22–402. 11

Speusippus, the son of Potone, sister of Plato, and his successor in the Academy before Xenocrates, was always full of zeal for the teachings of the Pythagoreans, and especially for the writings of Philolaus,

both odd and even. For this question, as well as many others arising in Greek arithmetic, the student may profitably consult *Nicomachus of Gerasa: Introduction to Arithmetic*, translated by Martin Luther D'Ooge, with studies in Greek arithmetic by Frank Egleston Robbins and Louis Charles Karpinski.

[b] *i.e.* both odd or both even.

[c] *i.e.* an odd number can be divided only into an odd number and an even number, never into two odd or two even numbers.

Φιλολάου συγγραμμάτων, βιβλίδιόν τι συντάξας γλαφυρὸν ἐπέγραψε μὲν αὐτὸ Περὶ Πυθαγορικῶν ἀριθμῶν, ἀπ' ἀρχῆς δὲ μέχρι ἡμίσους περὶ τῶν ἐν αὐτοῖς γραμμικῶν ἐμμελέστατα διεξελθὼν πολυγωνίων τε καὶ παντοίων τῶν ἐν ἀριθμοῖς ἐπιπέδων ἅμα καὶ στερεῶν, περί τε τῶν πέντε σχημάτων,[a] ἃ τοῖς κοσμικοῖς ἀποδίδοται στοιχείοις, ἰδιότητός ⟨τε⟩[1] αὐτῶν καὶ πρὸς ἄλληλα κοινότητος, ⟨περὶ⟩[2] ἀναλογίας τε καὶ ἀντακολουθίας,[3] μετὰ ταῦτα λοιπὸν θάτερον [τὸ][4] τοῦ βιβλίου ἥμισυ περὶ δεκάδος ἄντικρυς ποιεῖται, φυσικωτάτην αὐτὴν ἀποφαίνων καὶ τελεστικωτάτην τῶν ὄντων, οἷον εἶδός τι τοῖς κοσμικοῖς ἀποτελέσμασι τεχνικὸν ἀφ' ἑαυτῆς (ἀλλ' οὐχ ἡμῶν νομισάντων ἢ ὡς ἔτυχε) θεμέλιον ὑπάρχουσαν καὶ παράδειγμα παντελέστατον τῷ τοῦ παντὸς ποιητῇ θεῷ προεκκειμένην. λέγει δὲ τὸν τρόπον τοῦτον περὶ αὐτῆς.

"Ἔστι δὲ τὰ δέκα τέλειος ⟨ἀριθμός⟩,[5] καὶ ὀρθῶς τε καὶ κατὰ φύσιν εἰς τοῦτον καταντῶμεν παντοίως ἀριθμοῦντες Ἕλληνές τε καὶ πάντες ἄνθρωποι οὐδὲν αὐτοὶ ἐπιτηδεύοντες· πολλὰ γὰρ ἴδια ἔχει, ἃ προσήκει τὸν οὕτω τέλειον ἔχειν, πολλὰ δὲ ἴδια μὲν οὐκ ἔστιν αὐτοῦ, δεῖ δὲ ἔχειν αὐτὰ τέλειον.

"Πρῶτον μὲν οὖν ἄρτιον δεῖ εἶναι, ὅπως ἴσοι ἐνῶσιν οἱ περιττοί τε καὶ ἄρτιοι, καὶ μὴ ἑτερομερῶς[6]· ἐπεὶ γὰρ πρότερος ἀεί ἐστιν ὁ περιττὸς τοῦ

[1] ⟨τε⟩ add. Diels. [2] ⟨περὶ⟩ add. de Falco.
[3] ἀντακολουθίας Lang: ἀνακολουθίας Ast, Tannery, Diels.
[4] [τὸ] om. Diels.
[5] ἀριθμός add. Diels. [6] ἑτερομερεῖς Diels.

[a] For the five cosmic or Platonic figures, see *infra*, pp. 216-225.

and he compiled a neat little book which he entitled *On the Pythagorean Numbers*. From the beginning up to half way he deals most elegantly with linear and polygonal numbers and with all the kinds of surfaces and solids in numbers; with the five figures which he attributes to the cosmic elements,[a] both in respect of their special properties and in respect of their similarity one to another; and with proportion and reciprocity.[b] After this he immediately devotes the other half of the book to the decad, showing it to be the most natural and most initiative of realities, inasmuch as it is in itself (and not because we have made it so or by chance) an organizing idea of cosmic events, being a foundation stone and lying before God the Creator of the universe as a pattern complete in all respects. He speaks about it to the following effect.

"Ten is a perfect number, and it is both right and according to Nature that we Greeks and all men arrive at this number in all kinds of ways when we count, though we make no effort to do so; for it has many special properties which a number thus perfect ought to have, while there are many characteristics which, while not special to it, are necessary to its perfection.

"In the first place it must be even, in order that the odds and evens in it may be equal and not disparate. For since the odd is always prior to the even, unless

[b] If, with Ast, Tannery and Diels we read *ἀνακολουθίας* for *ἀντακολουθίας*, the rendering is "proportion continuous and discontinuous," but it is not easy to interpret this, though Tannery makes a valiant effort to do so. His French translation, notes and comments should be studied (*Pour l'histoire de la science hellène*, 2nd ed., pp. 374 *seq.*, 386 *seq.*, and *Mémoires scientifiques*, vol. i. pp. 281-289).

ἀρτίου, εἰ μὴ ἄρτιος εἴη ὁ συμπεραίνων, πλεονεκτήσει ὁ ἕτερος.

"Εἶτα δὲ ἴσους ἔχειν χρὴ τοὺς πρώτους καὶ ἀσυνθέτους καὶ τοὺς δευτέρους καὶ συνθέτους· ὁ δὲ δέκα ἔχει ἴσους, καὶ οὐδεὶς ἂν ἄλλος ἐλάττων τῶν δέκα τοῦτο ἔπαθεν ἀριθμός, πλείων δὲ τάχα (καὶ γὰρ ὁ ιβ̅ καὶ ἄλλοι τινές), ἀλλὰ πυθμὴν αὐτῶν ὁ δέκα· καὶ πρῶτος τοῦτο ἔχων καὶ ἐλάχιστος τῶν ἐχόντων τέλος τι ἔχει, καὶ ἴδιόν πως αὐτοῦ τοῦτο γέγονε τὸ ἐν πρώτῳ αὐτῷ ἴσους ἀσυνθέτους τε καὶ συνθέτους ὦφθαι.

"Ἔχων τε τοῦτο ἔχει πάλιν ⟨ἴσους⟩[1] καὶ τοὺς πολλαπλασίους καὶ τοὺς ὑποπολλαπλασίους, ὧν εἰσι πολλαπλάσιοι· ἔχει μὲν γὰρ ὑποπολλαπλασίους τοὺς μεχρὶ πέντε, τοὺς δὲ ἀπὸ τῶν ἓξ μέχρι τῶν δέκα [οἱ][2] πολλαπλασίους αὐτῶν· ἐπεὶ δὲ τὰ ζ̅ οὐδενός, ἐξαιρετέον, καὶ τὰ δ̅ ὡς πολλαπλάσια τοῦ β̅, ὥστε ἴσους εἶναι πάλιν [δεῖ].[3]

"Ἔτι πάντες οἱ λόγοι ἐν τῷ ι̅, ὅ τε τοῦ ἴσου καὶ τοῦ μείζονος καὶ τοῦ ἐλάττονος καὶ τοῦ ἐπι-

[1] ἴσους add. Lang.

[2] οἱ om. Diels.

[3] δεῖ om. Diels. He points out that the original reading may have been δ′, indicating the fourth property of the decad.

---

[a] One of the most noteworthy features of this passage is the early use of the terms *πρῶτοι καὶ ἀσύνθετοι* (*prime and incomposite*), *δεύτεροι καὶ σύνθετοι* (*secondary and composite*), for which see *supra*, p. 69 n. *c*. The use is different from that of Nicomachus and Iamblichus. It seems that *prime and incomposite* numbers are prime numbers in the ordinary sense, including 2, as is the case with Euclid and Aristotle (*Topics* Θ 2, 157 a 39). *Secondary and composite* numbers

the even were joined with it the other would predominate.

"Next it is necessary that the prime and incomposite and the secondary and composite [a] should be equal; now they are equal in the case of 10, and in the case of no other number which is less than 10 is this true, though numbers greater than 10 having this property (such as 12 and certain others [b]) can soon be found, but their base is 10. As the first number with this property and the least of those possessing it 10 has a certain perfection, and it is a property peculiar to itself that it is the first number in which the incomposite and the composite are equal.

"In addition to this property it has an equal number of multiples and submultiples of those multiples; for it has as submultiples the numbers up to 5, while those from 6 to 10 are multiples of them; since 7 is a multiple of no number, it has to be omitted, but 4 must also be dropped as a multiple of 2, and so this brings about equality once more.[c]

"Furthermore all the ratios are in 10, for the equal and the greater and the less and the superparticular

are all composite numbers, the term not being limited to odd numbers as with Nicomachus. There is no suggestion of a third mixed class. The two equal classes according to Speusippus are 1, 2, 3, 5, 7 and 4, 6, 8, 9, 10. According to the later terminology the *prime and incomposite* numbers would be 3, 5, 7, while the only *secondary and composite* number would be 9.

[b] Actually 10, 12 and 14 are the only numbers possessing this property.

[c] In the series 1, 2 . . . 10 the submultiples are 1, 2, 3, 5 and the multiples are 6, 8, 9, 10. It is curious that though 1 is counted as a submultiple, all the other numbers are not counted as multiples of it: to have admitted them as such would have destroyed the scheme.

μορίου καὶ τῶν λοιπῶν εἰδῶν ἐν αὐτῷ, καὶ οἱ γραμμικοὶ ⟨καὶ⟩[1] οἱ ἐπίπεδοι καὶ οἱ στερεοί. τὸ μὲν γὰρ ᾱ στιγμή, τὰ δὲ β̄ γραμμή, τὰ δὲ γ̄ τρίγωνον, τὰ δὲ δ̄ πυραμίς· ταῦτα δὲ πάντα ἐστὶ πρῶτα καὶ ἀρχαὶ τῶν καθ' ἕκαστον ὁμογενῶν. καὶ ἀναλογιῶν δὲ πρώτη αὕτη ἐστὶν ἡ ἐν αὐτοῖς ὀφθεῖσα, ἡ τὸ ἴσον μὲν ὑπερέχουσα, τέλος δὲ ἔχουσα ἐν τοῖς δέκα. ἔν τε ἐπιπέδοις καὶ στερεοῖς πρῶτά ἐστι ταῦτα, στιγμή, γραμμή, τρίγωνον, πυραμίς· ἔχει δὲ ταῦτα τὸν τῶν δέκα ἀριθμὸν καὶ τέλος ἴσχει. τετρὰς μὲν γὰρ ἐν πυραμίδος γωνίαις ἢ βάσεσιν, ἑξὰς δὲ ἐν πλευραῖς, ὥστε δέκα· τετρὰς δὲ πάλιν ἐν στιγμῆς καὶ γραμμῆς διαστήμασι καὶ πέρασι, ἑξὰς δὲ ἐν τριγώνου πλευραῖς καὶ γωνίαις, ὥστε πάλιν δέκα. καὶ μὴν καὶ ἐν τοῖς σχήμασι κατ' ἀριθμὸν σκεπτομένῳ συμβαίνει[2]· πρῶτον γάρ ἐστι τρίγωνον τὸ ἰσόπλευρον, ὃ ἔχει μίαν πως

[1] καὶ add. Lang.
[2] ⟨ταὐτὸ⟩ συμβαίνει Lang (in adn.), de Falco.

---

[a] Speusippus asserts that among the numbers 1, 2 . . . 10 all the different kinds of ratio can be found. The *superparticular* ratio is the ratio of the whole + an aliquot fraction, $1+\frac{1}{n}$ or $\frac{n+1}{n}$, typified by the ratio known as *ἐπίτριτος*, or $\frac{4}{3}$. Tannery sees here an allusion to the ten kinds of proportion outlined by Nicomachus (see *infra*, pp. 114-124), and a proof of their ancient origin.

[b] *i.e.*, 1, 2, 3, 4 form an arithmetical progression having 1 as the common difference and 10 as the sum.

[c] *i.e.*, a pyramid has 4 angles (or 4 faces) and 6 sides, and so exhibits the number 10.

[d] The reasoning is not very clear. Taking first a line and a point outside it, Speusippus notes that the line has 2 extremities and between the point and these 2 extremities are

and the remaining varieties are in it,[a] and so are the linear and plane and solid numbers. For 1 is a point, 2 is a line, 3 is a triangle and 4 is a pyramid; all these are elements and principles of the figures like to them. In these numbers is seen the first of the progressions, that in which the terms exceed by an equal amount, and they have 10 for their sum.[b] In surfaces and solids these are the elements—point, line, triangle, pyramid. The number 10 exhibits them and possesses perfection. For 4 is to be found in the angles or faces of a pyramid, and 6 in the sides,[c] so making 10; again 4 is to be found in the intervals and extremities of the point and line, while 6 is in the sides and angles of a triangle,[d] so as again to make 10. This also comes about in figures regarded from the point of view of number.[e] For the first triangle is the equilateral, which has one side and angle; I say one

2 intervals. This gives the number 4. A triangle has 3 sides and 3 angles, giving the number 6. Combining the point, the line and the triangle we thus get 10.

[e] A very difficult passage follows, but Tannery seems successfully to have unravelled its meaning. There seems to be here, he notes, an ill-developed Pythagorean conception. The point or monad is necessarily simple. The line is a dyad with two species, straight and curved. The triangle is a triad with three kinds. The pyramid is a tetrad with four kinds. Clearly the three species of triangle are the equilateral, the isosceles and the scalene, where the number of different elements are respectively 1, 2, 3. Speusippus does not consider isosceles and scalene triangles in general, but takes particular cases, and it is worthy of note that the three triangles he considers are used in the *Timaeus* of Plato.

By analogy, the pyramids can be divided into four kinds: (1) all solid angles equal; (2) three solid angles equal; (3) two solid angles equal; (4) all solid angles unequal. Here again Speusippus takes special cases, but he goes astray by giving the second class a square base, and has to force the analogy.

γραμμὴν καὶ γωνίαν· λέγω δὲ μίαν, διότι ἴσας ἔχει· ἄσχιστον γὰρ ἀεὶ καὶ ἑνοειδὲς τὸ ἴσον· δεύτερον δὲ τὸ ἡμιτετράγωνον· μίαν γὰρ ἔχον παραλλαγὴν γραμμῶν καὶ γωνιῶν ἐν δυάδι ὁρᾶται· τρίτον δὲ τὸ τοῦ ἰσοπλεύρου ἥμισυ τὸ καὶ ἡμιτρίγωνον· πάντως γὰρ ἄνισον καθ' ἕκαστον, τὸ δὲ πάντῃ[1] αὐτοῦ τρία ἐστί. καὶ ἐπὶ τῶν στερεῶν εὑρίσκοις ἂν ἄχρι τῶν τεττάρων προϊὸν τὸ τοιοῦτο, ὥστε δεκάδος καὶ οὕτως ψαύει· γίνεται γάρ πως ἡ μὲν πρώτη πυραμὶς μίαν πως γραμμήν τε καὶ ἐπιφάνειαν ἐν ἰσότητι ἔχουσα, ἐπὶ τοῦ ἰσοπλεύρου ἱσταμένη· ἡ δε δευτέρα δύο, ἐπὶ[2] τετραγώνου ἐνηγερμένη, μίαν παραλλαγὴν ἔχουσα[2] παρὰ τῆς ἐπὶ τῆς βάσεως γωνίας, ὑπὸ τριῶν ἐπιπέδων περιεχομένη, τὴν κατὰ κορυφὴν ὑπὸ τεττάρων συγκλειομένη, ὥστε ἐκ τούτου δυάδι ἐοικέναι· ἡ δὲ τρίτη τριάδι, ἐπὶ ἡμιτετραγώνου βεβηκυῖα καὶ σὺν τῇ ὀφθείσῃ μιᾷ ὡς ἐν ἐπιπέδῳ τῇ ἡμιτετραγώνῳ ἔτι καὶ ἄλλην ἔχουσα διαφορὰν τὴν τῆς κορυφαίας γωνίας, ὥστε τριάδι ἂν ὁμοιοῖτο, πρὸς ὀρθὰς τὴν γωνίαν ἔχουσα τῇ τῆς βάσεως μέσῃ πλευρᾷ· τετράδι δὲ ἡ τετάρτη κατὰ ταὐτά, ἐπὶ ἡμιτριγώνῳ[3] βάσει συνισταμένη, ὥστε τέλος ἐν τοῖς δέκα λαμβάνειν τὰ λεχθέντα. τὰ αὐτὰ δὲ καὶ ἐν τῇ γενέσει· πρώτη μὲν γὰρ ἀρχὴ εἰς μέγεθος στιγμή, δευτέρα γραμμή, τρίτη ἐπιφάνεια, τέταρτον στερεόν."

[1] πάντῃ Lang, de Falco; πᾶν [τι] Diels; Lang would like to read τὰ δὲ πάντα.

[2] ἐπὶ . . . ἔχουσα. Only one manuscript has these words; many emendations have been offered.

[3] The manuscripts have ἡμιτετραγώνῳ, but ἡμιτριγώνῳ is required, as Tannery recognized.

because they are equal; for the equal is always indivisible and uniform. The second triangle is the half-square; for with one difference in the sides and angles it corresponds to the dyad. The third is the half-triangle, which is half of the equilateral triangle; for being completely unequal in every respect, its elements number three. In the case of solids, you would find this property also, but going up to four, so that the decad is reached in this way also. For the first pyramid, which is built upon an equilateral triangle, is in some sense unity, since by reason of its equality it has one side and one face; the second pyramid, which is raised upon a square, has the angles at the base enclosed by three planes and that at the vertex by four, so that from this difference it resembles the dyad. The third resembles a triad, for it is set upon a half-square; together with the one difference that we have seen in the half-square as a plane figure it presents another corresponding to the angle at the vertex; there is therefore a resemblance between the triad and this pyramid, whose vertex lies on the perpendicular to the middle of the hypotenuse[a] of the base. In the same way the fourth, rising upon a half-triangle as base, resembles a tetrad, so that the aforesaid figures find completion in the number 10. The same result is seen in their generation. For the first principle of magnitude is point, the second is line, the third is surface, the fourth is solid."[b]

[a] Lit. "side."

[b] The abrupt end suggests that the passage went on in this strain for some time; but the historian of mathematics need not feel much disappointment.

Theon Smyr., ed. Hiller 45. 9–46. 19

Ἔτι τε τῶν ἀριθμῶν οἱ μέν τινες τέλειοι λέγονται, οἱ δ' ὑπερτέλειοι, οἱ δ' ἐλλιπεῖς. καὶ τέλειοι μέν εἰσιν οἱ τοῖς αὑτῶν μέρεσιν ἴσοι, ὡς ὁ τῶν ϛ̅· μέρη γὰρ αὐτοῦ ἥμισυ γ̅, τρίτον β̅, ἕκτον α̅, ἅτινα συντιθέμενα ποιεῖ τὸν ϛ̅. γεννῶνται δὲ οἱ τέλειοι τοῦτον τὸν τρόπον. ἐὰν ἐκθώμεθα τοὺς ἀπὸ μονάδος διπλασίους καὶ συντιθῶμεν αὐτούς, μέχρις οὗ ἂν γένηται πρῶτος καὶ ἀσύνθετος ἀριθμός, καὶ τὸν ἐκ τῆς συνθέσεως ἐπὶ τὸν ἔσχατον τῶν συντιθεμένων πολλαπλασιάσωμεν, ὁ ἀπογεννηθεὶς ἔσται τέλειος. οἷον ἐκκείσθωσαν διπλάσιοι α̅ β̅ δ̅ η̅ ιϛ̅. συνθῶμεν οὖν α̅ καὶ β̅· γίνεται γ̅· καὶ τὸν γ̅ ἐπὶ τὸν ὕστερον τὸν ἐκ τῆς συνθέσεως πολλαπλασιάσωμεν, τουτέστιν ἐπὶ τὸν β̅· γίνεται ϛ̅, ὅς ἐστι πρῶτος τέλειος. ἂν πάλιν τρεῖς τοὺς ἑφεξῆς διπλασίους συνθῶμεν, α̅ καὶ β̅ καὶ δ̅, ἔσται ζ̅· καὶ τοῦτον ἐπὶ τὸν ἔσχατον τῶν τῆς συνθέσεως πολλαπλασιάσωμεν, τὸν ζ̅ ἐπὶ τὸν δ̅· ἔσται ὁ κη̅, ὅς ἐστι δεύτερος τέλειος. σύγκειται ἐκ τοῦ ἡμίσεος τοῦ ιδ̅, τετάρτου τοῦ ζ̅, ἑβδόμου τοῦ δ̅, τεσσαρακαιδεκάτου τοῦ β̅, εἰκοστοῦ ὀγδόου τοῦ α̅.

Ὑπερτέλειοι δέ εἰσιν ὧν τὰ μέρη συντεθέντα μείζονά ἐστι τῶν ὅλων, οἷον ὁ τῶν ιβ̅· τούτου γὰρ ἥμισύ ἐστιν ϛ̅, τρίτον δ̅, τέταρτον γ̅, ἕκτον β̅, δωδέκατον α̅, ἅτινα συντεθέντα γίνεται ιϛ̅, ὅς ἐστι μείζων τοῦ ἐξ ἀρχῆς, τουτέστι τῶν ιβ̅.

Ἐλλιπεῖς δέ εἰσιν ὧν τὰ μέρη συντεθέντα ἐλάττονα τὸν ἀριθμὸν ποιεῖ τοῦ ἐξ ἀρχῆς προτεθέντος

[a] In other words, if $S_n = 1 + 2 + 2^2 + \ldots + 2^{n-1}$, and S_n is prime, then $S_n \cdot 2^{n-1}$ is a perfect number. This is proved in

Theon of Smyrna, ed. Hiller 45. 9–46. 19

Furthermore certain numbers are called *perfect*, some *over-perfect*, others *deficient*. Perfect numbers are those that are equal to their own parts, such as 6; for its parts are the half 3, the third 2 and the sixth 1, which added together make 6. Perfect numbers are produced in this manner. If we take successive double numbers starting from the unit and add them until a prime and incomposite number is found, and then multiply the sum by the last of the added terms, the resulting number will be perfect.[a] For example, let the doubles be 1, 2, 4, 8, 16. We therefore add together 1 and 2; the result is 3; and we multiply 3 by the last of the added terms, that is by 2; the result is 6, which is the first perfect number. Again, if we add together three doubles in order, 1 and 2 and 4, the result will be 7; and we multiply this by the last of the added terms, that is, we multiply 7 by 4; the result will be 28, which is the second perfect number. It is composed out of its half 14, its fourth part 7, its seventh part 4, its fourteenth part 2 and its twenty-eighth part 1.

Over-perfect numbers are those whose parts added together are greater than the wholes, such as 12; for the half of this number is 6, the third is 4, the fourth is 3, the sixth is 2 and the twelfth 1, which added together produce 16, and this is greater than the original number, 12.

Deficient numbers are those whose parts added together make a number less than the one originally

[a] Euclid ix. 36. Even the algebraic proof is too long for reproduction here, but for such a proof the reader may be referred to Heath, *The Thirteen Books of Euclid's Elements*, vol. ii. pp. 424–425.

ἀριθμοῦ, οἷον ὁ τῶν $\bar{\eta}$· τούτου γὰρ ἥμισυ $\bar{\delta}$, τέταρτον $\bar{\beta}$, ὄγδοον ἕν· τὸ αὐτὸ δὲ καὶ τῷ $\bar{\iota}$ συμβέβηκεν, ὃν καθ' ἕτερον λόγον τέλειον ἔφασαν οἱ Πυθαγορικοί, περὶ οὗ κατὰ τὴν οἰκείαν χώραν ἀποδώσομεν. λέγεται δὲ καὶ ὁ $\bar{\gamma}$ τέλειος, ἐπειδὴ πρῶτος ἀρχὴν καὶ μέσα καὶ πέρας ἔχει· ὁ δ' αὐτὸς καὶ γραμμή ἐστι καὶ ἐπίπεδον, τρίγωνον γὰρ ἰσόπλευρον ἑκάστην πλευρὰν δυεῖν μονάδων ἔχον, καὶ πρῶτος δεσμὸς καὶ στερεοῦ δύναμις· ἐν γὰρ τρισὶ διαστάσεσι τὸ στερεὸν νοεῖσθαι.

(d) Figured Numbers [a]

(i.) *General*

Nicom. *Arith. Introd.* ii. 7. 1-3, ed. Hoche 86. 9–87. 6

Ἔστιν οὖν σημεῖον ἀρχὴ διαστήματος, οὐ διάστημα δέ, τὸ δ' αὐτὸ καὶ ἀρχὴ γραμμῆς, οὐ γραμμὴ

---

[a] There were in use among the Greeks two ways of representing numbers geometrically. One, used by Euclid and implied in Plato, *Theaetetus* 147 D—148 B (see *infra*, p. 380), is to represent numbers by straight lines proportional in length to the numbers they represent. If two such lines are made adjacent sides of a rectangle, then the rectangle represents their product: if three such lines are made sides of a rectangular parallelepiped then the parallelepiped is the product. The other way of representing numbers was by dots or alphas for the units disposed along straight lines so as to form geometrical patterns, a method greatly developed by the Pythagoreans. Any number could be represented as a straight line, and prime numbers only as

put forth, such as 8; for the half of this number is 4, the fourth 2, the eighth 1. The same property is shown by 10, which the Pythagoreans called perfect for a different reason, and this we shall discuss in the proper place. The number 3 is also called perfect, since it is the first number which has a beginning and middle and end. It is moreover both a line and a surface, for it is an equilateral triangle in which each side is two units, and it is the first bond and power of the solid; for in three dimensions is the solid conceived.

(*d*) Figured Numbers [a]

(i.) *General*

Nicomachus, *Introduction to Arithmetic* ii. 7. 1-3, ed. Hoche 86. 9–87. 6

Point is therefore the principle of dimension, but is not dimension, while it is also the principle of line,

straight lines, whence Thymaridas spoke of them as "rectilinear *par excellence*" (Plato would have represented a prime number such as 7 by 7 × 1, an oblong). The unit, being the source of all number, can be taken as a triangle, a pentagon, a hexagon, and so on. The first number after 1 which can be represented as a triangle is 3, and the sum of the first *n* natural numbers can always be represented as a triangle; the adjoining figure, a famous Pythagorean symbol, shows how this is done for 1 + 2 + 3 + 4 = 10.

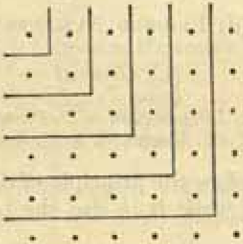
α
α α
α α α
α α α α

Square numbers can be represented in similar fashion, and the square of side *n* + 1 can be obtained from the square of side *n* by adding a gnomon of 2*n* + 1 dots round the side (the term "gnomon" originally signified an upright stick which cast shadows on a plane or hemispherical surface, and so

δέ· καὶ γραμμὴ ἀρχὴ ἐπιφανείας, οὐκ ἐπιφάνεια δέ, καὶ ἀρχὴ τοῦ διχῇ διαστατοῦ, οὐ διχῇ δὲ διαστατόν. καὶ εἰκότως ἡ ἐπιφάνεια ἀρχὴ μὲν σώματος, οὐ σῶμα δέ, καὶ ἡ αὐτὴ ἀρχὴ μὲν τοῦ τριχῇ διαστατοῦ, οὐ τριχῇ δὲ διαστατόν. οὕτως δὴ καὶ ἐν τοῖς ἀριθμοῖς ἡ μὲν μονὰς ἀρχὴ παντὸς ἀριθμοῦ ἐφ' ἓν διάστημα κατὰ μονάδα προβιβαζομένου, ὁ δὲ γραμμικὸς ἀριθμὸς ἀρχὴ ἐπιπέδου ἀριθμοῦ ἐφ' ἕτερον διάστημα ἐπιπέδως πλατυνομένου, ὁ δὲ ἐπίπεδος ἀριθμὸς ἀρχὴ στερεοῦ ἀριθμοῦ ἐπὶ τρίτον

---

could be used for telling the time; it was later used of an

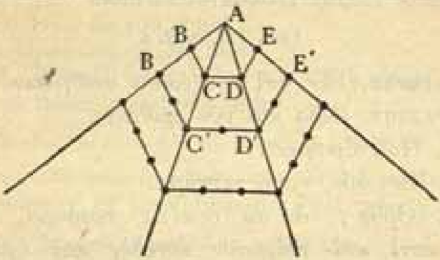


instrument for drawing right angles).

The first number after 1 which can be represented as a pentagon is 5. If it be represented as ABCDE, then we can form another pentagon AB′C′D′E, equivalent to 10, by adding the "gnomon of the pentagon," a row of an extra 7 dots arranged round three of the sides of the original pentagon. The gnomons to be added to form the successive pentagonal numbers 1, 5, 12, 22 . . . are respectively 4, 7, 10 . . ., or the successive terms of an arithmetical progression having 3 as the common difference. In the case of the hexagon the successive gnomonic numbers differ by 4, and in general, if n is the number of sides in the polygon, the successive gnomonic numbers differ by $n-2$.

but is not line ; and line is the principle of surface, but is not surface, and is the principle of the two-dimensional, but is not two-dimensional. Naturally also surface is the principle of body, but is not body, while it is the principle of the three-dimensional, but is not three-dimensional. Similarly among numbers the unit is the principle of every number set out by units in one dimension, while linear number is the principle of plane number broadened out in another dimension in the manner of a surface, and plane number is the principle of solid number, which acquires a certain depth in a third dimension [at

---



So much for plane numbers. There are similar varieties of solid numbers (cubes, pyramids, truncated pyramids, etc.). The curious reader will find the whole subject treated exhaustively by Nicomachus (*Arith. Introd.* ii. 7-20), Theon of Smyrna (ed. Hiller 26-42) and Iamblichus (*in Nicom. Arith. Introd.*, ed. Pistelli 58. 7 *et seq.*). It is of importance for the student of Greek mysticism, but has little interest for the modern mathematician.

διάστημα πρὸς τὰ ἐξ ἀρχῆς βάθος τι προσκτωμένου· οἷον καθ' ὑποδιαίρεσιν γραμμικοὶ μέν εἰσιν ἀριθμοὶ ἁπλῶς ἅπαντες οἱ ἀπὸ δυάδος ἀρχόμενοι καὶ κατὰ μονάδος πρόσθεσιν ἐπὶ ἓν καὶ τὸ αὐτὸ προχωροῦντες διάστημα, ἐπίπεδοι δὲ οἱ ἀπὸ τριάδος ἀρχόμενοι ἀρχικωτάτης ῥίζης καὶ διὰ τῶν ἑξῆς συνεχῶν ἀριθμῶν προϊόντες, λαμβάνοντες καὶ τὴν ἐπωνυμίαν κατὰ τὴν αὐτὴν τάξιν· πρώτιστοι γὰρ τρίγωνοι, εἶτα μετ' αὐτοὺς τετράγωνοι, εἶτα μετ' αὐτοὺς πεντάγωνοι, εἶτα ἐπὶ τούτοις ἑξάγωνοι καὶ ἑπτάγωνοι καὶ ἐπ' ἄπειρον.

(ii.) *Triangular Numbers*

Luc. *Vit. auct.* 4

ΠΥΘΑΓΟΡΑΣ. Εἶτ' ἐπὶ τουτέοισιν ἀριθμέειν.

ΑΓΟΡΑΣΤΗΣ. Οἶδα καὶ νῦν ἀριθμεῖν.

ΠΥΘ. Πῶς ἀριθμέεις;

ΑΓΟ. Ἕν, δύο, τρία, τέτταρα.

ΠΥΘ. Ὁρᾷς; ἃ σὺ δοκέεις τέσσαρα, ταῦτα δέκα ἐστὶ καὶ τρίγωνον ἐντελὲς καὶ ἡμέτερον ὅρκιον.

Procl. *in Eucl.* i., ed. Friedlein 428. 7–429. 8

Παραδέδονται δὲ καὶ μέθοδοί τινες τῆς εὑρέσεως τῶν τοιούτων τριγώνων, ὧν τὴν μὲν εἰς Πλάτωνα

[a] This celebrated Pythagorean symbol was known as the

right angles] to the dimensions of the surface. For example, by subdivision linear numbers are all numbers without exception beginning from two and proceeding by the addition of a unit in one and the same dimension, while plane numbers begin from three as their fundamental root and advance through an orderly series of numbers, taking their designation according to their order. For first come triangles, then after them are squares, then after these are pentagons, then succeeding these are hexagons and heptagons and so on to infinity.

(ii.) *Triangular Numbers*

Lucian, *Auction of Souls* 4

PYTHAGORAS. After this you must count.

AGORASTES. Oh, I know how to do that already.

PYTH. How do you count ?

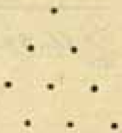
AGO. One, two, three, four.

PYTH. Do you see ? What you think is four is ten, a perfect triangle and our oath.[a]

Proclus, *on Euclid* i., ed. Friedlein 428. 7–429. 8

There have been handed down certain methods for the discovery of such triangles,[b] of which one is

τετρακτύς. It was alternatively called the "principle of health" (Lucian, *De Lapsu in Salutando* 5). The sum of any number of successive terms (beginning with the first) of the series of natural numbers $1+2+3+\ldots+n$ is therefore a triangular number, and the general formula for a triangular number is $\frac{1}{2}n(n+1)$.



[b] *i.e.*, triangles having the square on one side equal to the sum of the squares on the other two. Proclus is commenting on Euclid i. 47, for which see *infra*, pp. 178-185.

ἀναπέμπουσι, τὴν δὲ εἰς Πυθαγόραν. καὶ ἡ μὲν Πυθαγορικὴ ἀπὸ τῶν περιττῶν ἐστιν ἀριθμῶν. τίθησι γὰρ τὸν δοθέντα περιττὸν ὡς ἐλάσσονα τῶν περὶ τὴν ὀρθήν, καὶ λαβοῦσα τὸν ἀπ' αὐτοῦ τετράγωνον καὶ τούτου μονάδα ἀφελοῦσα τοῦ λοιποῦ τὸ ἥμισυ τίθησι τῶν περὶ τὴν ὀρθὴν τὸν μείζονα· προσθεῖσα δὲ καὶ τούτῳ μονάδα τὴν λοιπὴν ποιεῖ τὴν ὑποτείνουσαν· οἷον τὸν τρία λαβοῦσα καὶ τετραγωνίσασα καὶ ἀφελοῦσα τοῦ ἐννέα μονάδα τοῦ η̅ λαμβάνει τὸ ἥμισυ τὸν δ̅, καὶ τούτῳ προστίθησι πάλιν μονάδα καὶ ποιεῖ τὸν ε̅, καὶ εὕρηται τρίγωνον ὀρθογώνιον ἔχον τὴν μὲν τριῶν, τὴν δὲ τεσσάρων, τὴν δὲ πέντε.

Ἡ δὲ Πλατωνικὴ ἀπὸ τῶν ἀρτίων ἐπιχειρεῖ. λαβοῦσα γὰρ τὸν δοθέντα ἄρτιον τίθησιν αὐτὸν ὡς μίαν πλευρὰν τῶν περὶ τὴν ὀρθήν, καὶ τοῦτον

---

[a] *i.e.*, if n is the given odd number, the sides of the triangle are

$$n, \frac{n^2-1}{2}, \frac{n^2+1}{2}$$

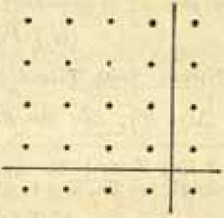
and the formula is an assertion that

$$n^2+\left(\frac{n^2-1}{2}\right)^2=\left(\frac{n^2+1}{2}\right)^2.$$

referred to Plato and one to Pythagoras. The Pythagorean method starts from the odd numbers. For it sets the given odd number as the lesser of the sides about the right angle, takes its square and subtracts a unit therefrom, and sets half the result as the greater of the sides about the right angle. Adding a unit to this it makes the resulting number the hypotenuse.[a] For example, starting from 3 and squaring, the method obtains 9; a unit is subtracted, making 8, and the half of 8 is taken, making 4; to this a unit is added, giving 5, and in this way there is found a right-angled triangle having as its respective sides 3, 4 and 5.

The Platonic method starts from the even numbers. For taking the given even number it sets it as one of the sides about the right angle, divides

[a] Heath (*H.G.M.* i. 80) shows how Pythagoras probably arrived at this formula by a system of dots forming a square. Starting with a square of side m, the square of side $m+1$ can be formed by adding a gnomon-like array of $2m+1$ dots round two sides. To obtain his formula, Pythagoras would only have to assume that $2m+1$ (necessarily an odd number) is a square.



Let $$2m+1=n^2$$
then $$m=\frac{n^2-1}{2}$$
$$m+1=\frac{n^2+1}{2}$$
and the array of dots shows that
$$n^2+\left(\frac{n^2-1}{2}\right)^2=\left(\frac{n^2+1}{2}\right)^2.$$

διελοῦσα δίχα καὶ τετραγωνίσασα τὸ ἥμισυ, μονάδα μὲν τῷ τετραγώνῳ προσθεῖσα ποιεῖ τὴν ὑποτείνουσαν, μονάδα δὲ ἀφελοῦσα τοῦ τετραγώνου ποιεῖ τὴν ἑτέραν τῶν περὶ τὴν ὀρθήν· οἷον τὸν τέσσαρα λαβοῦσα καὶ τούτου τὸ ἥμισυ τὸν β̄ τετραγωνίσασα καὶ ποιήσασα αὐτὸν δ̄. ἀφελοῦσα μὲν μονάδα ποιεῖ τὸν γ̄, προσθεῖσα δὲ ποιεῖ τὸν ε̄, καὶ ἔχει τὸ αὐτὸ γενόμενον τρίγωνον, ὃ καὶ ἐκ τῆς ἑτέρας ἀπετελεῖτο μεθόδου. τὸ γὰρ ἀπὸ τούτου ἴσον τῷ ἀπὸ τοῦ γ̄ καὶ τῷ ἀπὸ τοῦ δ̄ συντεθεῖσιν.

(iii.) *Oblong and Square Numbers*

Aristot. *Phys.* Γ 4, 203 a 13-15

Περιτιθεμένων γὰρ τῶν γνωμόνων περὶ τὸ ἓν καὶ χωρὶς ὁτὲ μὲν ἄλλο ἀεὶ γίγνεσθαι τὸ εἶδος, ὁτὲ δὲ ἕν.

(iv.) *Polygonal Numbers*

Nicom. *Arith. Introd.* ii. 12. 2-4, ed. Hoche 96. 11–97. 17

Δύο δή, οὓς ἂν θέλῃς, τριγώνους συνεχεῖς ἀλ-

---

[a] *i.e.*, if $2n$ is the given even number, the sides of the triangle are $2n$, n^2+1, n^2-1, and the formula asserts that

$$(2n)^2+(n^2-1)^2=(n^2+1)^2.$$

Heath (*H.G.M.* i. 81) shows how this formula, like that of Pythagoras, could have been obtained from gnomons of dots. Both formulae can be deduced from Euclid ii. 5, a Pythagorean proposition (see *infra*, p. 194 n. *a*). A more general formula, including both the Pythagorean and Platonic methods, is given in the lemma to Euclid x. 28, which is equivalent to the assertion

$$m^2n^2p^2q^2+\left(\frac{mnp^2-mnq^2}{2}\right)^2=\left(\frac{mnp^2+mnq^2}{2}\right)^2.$$

this in two and squares the half, adds a unit to the square so as to make the hypotenuse and subtracts a unit from the square so as to make the other side about the right angle.[a] For example, taking 4 and squaring the half, 2, it makes 4 again. Subtracting a unit it obtains 3, and adding one it makes 5, and yields the same triangle as that furnished by the other method. For the triangle constructed by this method is equal to that from 3 and from 4.

(iii.) *Oblong and Square Numbers*

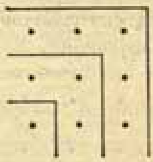
Aristotle, *Physics* Γ 4, 203 a 13-15

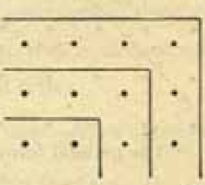
For when gnomons are placed round 1 the resulting figures are in one case always different, in the other they preserve one form.[b]

(iv.) *Polygonal Numbers*

Nicomachus, *Introduction to Arithmetic* ii. 12. 2-4, ed. Hoche 96. 11-97. 17

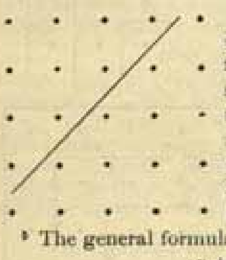
By taking any two successive triangular numbers





[b] As was indicated on p. 86 n. *a*, when gnomons consisting of an *odd* number of dots are placed round 1 the result is always a square. When gnomons consisting of an *even* number of dots are placed round 2 the result is an oblong, and the successive oblongs are always different in form. This is probably what Aristotle refers to, but he does not indicate that the starting-point is in one case 1 and in the other 2; and the interpretation is modern, Themistius and Simplicius having other (and less attractive) explanations. The subject is fully discussed by W. D. Ross in his notes *ad loc.* (*Aristotle's Physics*, pp. 542-544).

λήλοις συνθεὶς πάντως τετράγωνον ποιήσεις καὶ ὁντινοῦν τετράγωνον ἄρα διαλύσας δυνήσῃ δύο ἀπ' αὐτῶν τριγώνους ποιῆσαι· καὶ πάλιν παντὶ τετραγώνῳ σχήματι τρίγωνον προσζευχθὲν ὁθενοῦν πεντάγωνον ποιεῖ, οἷον τῷ δ̅ τετραγώνῳ ὁ α̅ τρίγωνος προσζευχθεὶς τὸν ε̅ πεντάγωνον ποιεῖ καὶ τῷ θ̅ τῷ ἑξῆς ὁ ἑξῆς προστεθείς, δηλονότι ὁ γ̅, πεντάγωνον τὸν ι̅β̅ ποιεῖ, τῷ δὲ ι̅ϛ̅ ὄντι ἀκολούθῳ ὁ ϛ̅ ἀκόλουθος ἐπισυντεθεὶς τὸν κ̅β̅ ἀκόλουθον ἀποδίδωσιν καὶ τῷ κ̅ε̅ ὁ ι̅ τὸν λ̅ε̅ καὶ ἀεὶ οὕτως. κατὰ δὲ τὰ αὐτὰ κἂν τοῖς πενταγώνοις οἱ τρίγωνοι προστιθοῖντο τῇ αὐτῇ τάξει, τοὺς εὐτάκτους γεννήσουσιν ἑξαγώνους καὶ πάλιν ἐκείνοις οἱ αὐτοὶ προσπλεκόμενοι τοὺς ἐν τάξει ἑπταγώνους ποιήσουσι καὶ μετ' ἐκείνους τοὺς ὀκταγώνους καὶ τοῦτο ἐπ' ἄπειρον. πρὸς δὲ ὑπόμνησιν ἐκκείσθωσαν ἡμῖν πολυγώνων στίχοι παραλλήλως γεγραμμένοι οἵδε, ὁ πρῶτος τριγώνων, ὁ μετ' αὐτὸν τετραγώνων, μετὰ δὲ ἀμφοτέρους πενταγώνων, εἶτα ἑξαγώνων, εἶτα ἑπταγώνων, εἶτα, εἰ ἐθέλοι τις, καὶ τῶν ἑξῆς πολυγώνων·



[a] In other words $\frac{1}{2}(n-1)n + \frac{1}{2}n(n+1) = n^2$, as may easily be seen from an array of dots. Here the square, of side n, is split up into two triangular numbers of side $n-1$, n whose values are therefore $\frac{1}{2}(n-1)n$, $\frac{1}{2}n(n+1)$. Theon of Smyrna (ed. Hiller 41. 3-8) gives the same theorem.

[b] The general formula for an a-gonal number of side n is
$$n + \tfrac{1}{2}n(n-1)(a-2).$$

you please and adding them one to another you will make the whole into a square, and whatsoever square you split up you will be able to make two triangles from it.[a] Again, a triangle joined to any square figure makes a pentagon ; for example, when the triangle 1 is added to the square 4 it makes the pentagon 5, and when the next triangle in order, which is plainly 3, is joined to 9, the next square, it makes 12, while 6, the next successive triangle, added to 16, the next successive square, will yield 22, the next successive pentagon, and 10 added to 25 will make 35, and so on without limit. In the same way if the triangles are added to the corresponding pentagons, they will produce the hexagons in an orderly series, and the triangles linked with them in turn will give the heptagons in order, and after them the octagons, and so on to infinity.[b] To help the memory let the various polygonal numbers be written out in parallel rows, the first consisting of triangles, the next of squares, the next after these of pentagons, then of hexagons, then of heptagons, then, if it is so desired, of the other polygonal numbers in order.

as is proved below, p. 98 n. *a*, and Nicomachus's assertion is equivalent to saying

$$n+\tfrac{1}{2}n(n-1)(a-2)=n+\tfrac{1}{2}n(n-1)(a-3)+\tfrac{1}{2}n(n-1).$$

μῆκος καὶ πλάτος

τρίγωνοι	α	γ	ϛ	ι	ιε	κα	κη	λϛ	με	νε	βάθος
τετράγωνοι	α	δ	θ	ιϛ	κε	λϛ	μθ	ξδ	πα	ρ	
πεντάγωνοι	α	ε	ιβ	κβ	λε	να	ο	ϙβ	ριζ	ρμε	
ἑξάγωνοι	α	ϛ	ιε	κη	με	ξϛ	ϙα	ρκ	ρνγ	ρϙ	
ἑπτάγωνοι	α	ζ	ιη	λδ	νε	πα	ριβ	ρμη	ρπθ	σλε	

(v.) *Gnomons of Polygonal Numbers*

Iambl. *in Nicom. Arith. Introd.*, ed. Pistelli 62. 10-18

Καὶ ἐν τῇ σχηματογραφίᾳ δὲ τῶν πολυγώνων δύο μὲν ἐπὶ πάντων αἱ αὐταὶ μενοῦσι πλευραὶ μηκυνόμεναι καθʼ ἕκαστον, αἱ δὲ παρὰ ταύτας ἐναποληφθήσονται τῇ τῶν γνωμόνων περιθέσει αἰεὶ ἀλλασσόμεναι, μία μὲν ἐν τριγώνῳ, δύο δὲ ἐν τετραγώνῳ καὶ τρεῖς ἐν πενταγώνῳ καὶ ὁμοίως ἐπʼ ἄπειρον, κατὰ δυάδος κἀνταῦθα διαφορὰν τῆς κλήσεως τῶν πολυγώνων πρὸς τὴν ποσότητα τῶν ἀλλασσομένων γινομένης.

---

[a] *i.e.*, the principle will be made clear from the figures for the gnomons of the square and pentagon given on pp. 86-89 n. *a*. The general formula is that in a polygon of *a* sides, the number of sides changed to form the next highest polygon

Breadth and Length

Triangles	1	3	6	10	15	21	28	36	45	55	
Squares	1	4	9	16	25	36	49	64	81	100	
Pentagons	1	5	12	22	35	51	70	92	117	145	depth
Hexagons	1	6	15	28	45	66	91	120	153	190	
Heptagons	1	7	18	34	55	81	112	148	189	235	

(v.) *Gnomons of Polygonal Numbers*

Iamblichus, *On Nicomachus's Introduction to Arithmetic*, ed. Pistelli 62. 10-18

Now in the representation of the polygons two of the sides always remain the same but are produced, while the sides intercepted between them are continually changed when the gnomons are placed round, one being changed in the triangle, two in the square, three in the pentagon and so on to infinity, the difference between the designation of the polygons and the number of sides changed being two.[a]

is $a-2$. (This leads Iamblichus to introduce immediately Thymaridas's rule for solving n simultaneous equations, as the factor $a-2$ occurs in this also. For this rule see *infra*, pp. 138-141).

From Iamblichus's account it follows that the successive gnomons to a polygon of a sides are

$$1,\ 1+(a-2),\ 1+2(a-2),\ .\ .\ .\ 1+(r-1)(a-2),$$

and the a-gonal number of side n is the sum of n terms this series, or

$$n+\tfrac{1}{2}n(n-1)(a-2).$$

(e) Some Properties of Numbers

(i.) *The "Sieve" of Eratosthenes*

Nicom. *Arith. Introd.* i. 13. 2-4, ed. Hoche 29. 17–32. 18

Ἡ δὲ τούτων γένεσις ὑπὸ Ἐρατοσθένους καλεῖται κόσκινον, ἐπειδὴ ἀναπεφυρμένους τοὺς περισσοὺς λαβόντες καὶ ἀδιακρίτους ἐξ αὐτῶν τῇ τῆς γενέσεως μεθόδῳ ταύτῃ διαχωρίζομεν, ὡς δι' ὀργάνου ἢ κοσκίνου τινὸς καὶ ἰδίᾳ μὲν τοὺς πρώτους καὶ ἀσυνθέτους, ἰδίᾳ δὲ τοὺς δευτέρους καὶ συνθέτους, χωρὶς δὲ τοὺς μικτοὺς εὑρίσκομεν. ἔστι δὲ ὁ τρόπος τοῦ κοσκίνου τοιοῦτος· ἐκθέμενος τοὺς ἀπὸ τριάδος πάντας ἐφεξῆς περισσοὺς ὡς δυνατὸν μάλιστα ἐπὶ μήκιστον στίχον, ἀρξάμενος ἀπὸ τοῦ πρώτου ἐπισκοπῶ, τίνας οἷός τέ ἐστι μετρεῖν, καὶ εὑρίσκω δυνατὸν ὄντα τοὺς δύο μέσους παραλείποντας μετρεῖν, μέχρις οὗ ἂν προχωρεῖν ἐθέλωμεν, οὐχ ὡς ἔτυχε δὲ καὶ εἰκῇ μετροῦντα, ἀλλὰ τὸν μὲν πρώτως κείμενον, τουτέστι τὸν δύο μέσους ὑπερβαίνοντα κατὰ τὴν τοῦ πρωτίστου ἐν τῷ στίχῳ κειμένου ποσότητα μετρήσει, τουτέστι κατὰ τὴν ἑαυτοῦ· τρὶς γάρ· τὸν δ' ἀπ' ἐκείνου δύο

[a] Nicomachus has been discussing the different species of odd numbers, which are explained above on p. 69 n. c.

[b] That is, Eratosthenes, for whom see p. 156 n. a, set out the odd numbers beginning with 3 in a column. For convenience we will set them out horizontally as follows:

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35.

(e) Some Properties of Numbers

(i.) *The "Sieve" of Eratosthenes*

Nicomachus, *Introduction to Arithmetic* i. 13. 2-4, ed. Hoche 29. 17–32. 18

The method of obtaining these[a] is called by Eratosthenes a sieve, since we take the odd numbers mixed together and indiscriminate, and out of them by this method, as though by some instrument or sieve, we separate the prime and incomposite by themselves, and the secondary and composite by themselves, and also find those that are mixed. The nature of the sieve is as follows : I set forth in as long a column as possible all the odd numbers, beginning with three, and, starting with the first, I examine which numbers in the series it will measure, and I find it will measure the numbers obtained by passing over two intermediate numbers, so far as we care to proceed, not measuring them at random and by haphazard, but it will measure the number first found by this process, that is, the one obtained by passing over two intermediate numbers, according to the magnitude of the number lying at the head of the column, that is, according to the magnitude of itself ; for it will measure it thrice.[b] It will measure the number

We now strike out from this list the multiples of 3, because they will not be prime numbers, and this is done by passing over two numbers at a time and striking out the next. That is, we pass over 5 and 7 and strike out 9, we pass over 11 and 13 and strike out 15, and so on without limit. As Nicomachus notes in a rather cumbrous way, the numbers struck out, 3, 9, 15, 21, 27 . . ., when divided by 3 gives us in order the numbers in the original column 3, 5, 7, 9 . . . . There is here the foundation for a logical theory of the infinite, but it was left for Russell and Whitehead to develop it.

διαλείποντα κατὰ τὴν τοῦ δευτέρου τεταγμένου· πεντάκις γάρ· τὸν δὲ περαιτέρω πάλιν δύο διαλείποντα κατὰ τὴν τοῦ τρίτου τεταγμένου· ἑπτάκις γάρ. τὸν δὲ ἔτι περαιτέρω ὑπὲρ δύο κείμενον κατὰ τὴν τοῦ τετάρτου τεταγμένου· ἐνάκις γάρ· καὶ ἐπ᾿ ἄπειρον τῷ αὐτῷ τρόπῳ. εἶτα μετὰ τοῦτον ἀπ᾿ ἄλλης ἀρχῆς ἐπὶ τὸν δεύτερον ἐλθὼν σκοπῶ, τίνας οἷός τέ ἐστι μετρεῖν, καὶ εὑρίσκω πάντας τοὺς τετράδα διαλείποντας, ἀλλὰ τὸν μὲν πρῶτον κατὰ τὴν τοῦ ἐν τῷ στίχῳ πρώτου τεταγμένου ποσότητα· τρὶς γάρ· τὸν δὲ δεύτερον κατὰ τὴν τοῦ δευτέρου· πεντάκις γάρ· τὸν δὲ τρίτον κατὰ τὴν τοῦ τρίτου· ἑπτάκις γάρ· καὶ τοῦτο ἐφεξῆς ἀεί.

(ii.) *Divisibility of Squares*

Theon Smyr., ed. Hiller 35. 17–36. 2

Ἰδίως δὲ τοῖς τετραγώνοις συμβέβηκεν ἤτοι τρίτον ἔχειν ἢ μονάδος ἀφαιρεθείσης τρίτον ἔχειν πάντως, ἢ πάλιν τέταρτον ἔχειν ἢ μονάδος ἀφαιρεθείσης τέταρτον ἔχειν πάντως· καὶ τὸν μὲν μονάδος ἀφαιρεθείσης τρίτον ἔχοντα ἔχειν καὶ τέταρτον

---

[a] The numbers obtained by passing over four numbers are

15, 25, 35 . . .

and can all be divided by 5, leaving

3, 5, 7 . . .

which is the original series of odd numbers.

Nicomachus proceeds to pass over six numbers at a time, beginning from 7, but we need not follow him. Clearly in this way he will eventually be able to remove from the series of odd numbers all that are not prime. The general formula is that we obtain all multiples of a prime number *n* by skip-

obtained by passing over two from that one according to the magnitude of the second number in order; for it will measure it five times. The number obtained by passing over two numbers yet again it will measure according to the magnitude of the third number in order; for it will measure it seven times. The number that lies yet two places beyond it will measure according to the magnitude of the fourth number in order; for it will measure it nine times; and we may proceed without limit in this manner. After this I make a fresh start with the second number in the series and examine which numbers it will measure, and I find it will measure all the numbers obtained by passing over four,[a] and will measure the first number so obtained according to the magnitude of the first number in the column; for it will measure it thrice. It will measure the second according to the magnitude of the second, that is, five times; the third according to the magnitude of the third, that is, seven times; and so on in order for ever.

(ii.) *Divisibility of Squares*

Theon of Smyrna, ed. Hiller 35. 17–36. 2

It is a property of squares to be divisible by three, or to become so divisible after subtraction of a unit; likewise they are divisible by four, or become so divisible after subtraction of a unit; even squares that after subtraction of a unit are divisible by three

ping $n-1$ terms at a time. But to make sure that any odd number $2n+1$ left in the series is prime we should have to try to divide it by all the prime numbers up to $\sqrt{2n+1}$, and the method is not a practicable way of ascertaining whether any large number is prime.

πάντως, ὡς ὁ δ̄, τὸν δὲ μονάδος ἀφαιρεθείσης τέταρτον ἔχοντα ἔχειν τρίτον πάντως, ὡς ὁ θ̄, ἢ τὸν αὐτὸν πάλιν καὶ τρίτον ἔχειν καὶ τέταρτον, ὡς ὁ λϛ̄ [ἢ μηδέτερον τούτων ἔχοντα τοῦτον μονάδος ἀφαιρεθείσης τρίτον ἔχειν πάντως],[1] ἢ μήτε τρίτον μήτε τέταρτον ἔχοντα μονάδος ἀφαιρεθείσης καὶ τρίτον ἔχειν καὶ τέταρτον, ὡς ὁ κε̄.

(iii.) *A Theorem about Cube Numbers*

Nicom. *Arith. Introd.* ii. 20. 5, ed. Hoche 119. 12-18

Ἐκτεθέντων γὰρ τῶν ἀπὸ μονάδος ἐπ' ἄπειρον συνεχῶν περισσῶν ἐπισκόπει οὕτως, ὁ πρῶτος τὸν δυνάμει κύβον ποιεῖ, οἱ δὲ δύο μετ' ἐκεῖνον συντεθέντες τὸν δεύτερον, οἱ δὲ ἐπὶ τούτοις τρεῖς τὸν τρίτον, οἱ δὲ συνεχεῖς τούτοις τέσσαρες τὸν τέταρτον, οἱ δὲ ἐφεξῆς τούτοις πέντε τὸν πέμπτον

[1] ἢ . . . πάντως om. Bullialdus, Hiller.

---

[a] Any number may be written as $3n$, $3n \pm 1$ or $3n \pm 2$, and its square takes the form

$$9n^2 \text{ or } 9n^2 \pm 6n + 1 \text{ or } 9n^2 \pm 12n + 4.$$

In the first case, the square is divisible by three; in the second and third cases it becomes so divisible after subtraction of a unit.

can be divided by four, such as 4 itself; those that after subtraction of a unit are divisible by four can be divided by three, such as 9; while there are yet again squares divisible both by three and by four, such as 36; and others that are divisible neither by three nor by four but can be divided, after subtraction of a unit, by both three and four, such as 25.[a]

(iii.) *A Theorem about Cube Numbers*

Nicomachus, *Introduction to Arithmetic* ii. 20. 5, ed. Hoche 119. 12-18

When the odd numbers beginning with one are set out in succession *ad infinitum* this property can be noticed, that the first makes a cube, the sum of the next two after it makes the second cube, the next three following them make the third cube, the next four succeeding these make the fourth cube, the next five in order after these makes the fifth cube,

[a] As for division by four, the square of an even number $2n$ is necessarily divisible by 4. The square of an odd number $2n \pm 1$ may be written $4n^2 \pm 4n + 1$ and becomes divisible by four after subtraction of a unit. Karpinski observes (*Nicomachus of Gerasa*, by M. L. D'Ooge, p. 58): "Apparently Theon desired to divide all square numbers into four classes, viz., those divisible by three and not by four; by four and not by three; by three and four; and by neither three nor four. In modern mathematical phraseology all square numbers are termed congruent to 0 or 1, modulus 3, and congruent to 0 or 1, modulus 4. This is written:

$$n^2 \equiv 1 \text{ (mod. 3)},$$
$$n^2 \equiv 0 \text{ (mod. 3)},$$
$$n^2 \equiv 0 \text{ or } 1 \text{ (mod. 4)}.$$

"This is the first appearance of any work on congruence which is fundamental in the modern theory of numbers."

καὶ οἱ ἑξῆς ἓξ τὸν ἕκτον καὶ τοῦτο μέχρις αἰεί.

(iv.) *A Property of the Pythmen*

Iambl. *in Nicom. Arith. Introd.*, ed. Pistelli 103. 10–104. 13

Ἐπεὶ δὲ ἑξάδος ἀποτελεστική ἐστιν ἡ πρώτη παρ' οὐδὲν ἀπὸ μονάδος συζυγία, ἡ πρώτη $\bar{α}$ $\bar{β}$ $\bar{γ}$ εἰδοποιήσει τὰς ἑξῆς αὐτῇ, μηδενὸς ὅρου κοινοῦ λαμβανομένου μηδὲ μὴν παρελλειπομένου, ἀλλὰ μετὰ τὴν $\bar{α}$ $\bar{β}$ $\bar{γ}$ λαμβανομένης τῆς $\bar{δ}$ $\bar{ε}$ $\bar{ϛ}$, εἶτα $\bar{ζ}$ $\bar{η}$ $\bar{θ}$ καὶ ἑξῆς ἀκολούθως. πᾶσαι γὰρ αὗται ἑξάδες γενήσονται μεταλαμβανούσης τὸν μονάδος τόπον ἀεὶ τῆς δεκάδος, τουτέστιν εἰς μονάδα ἀναγομένης· οὕτως γὰρ αὐτὴν καὶ δευτερωδουμέναν μονάδα καλεῖσθαι ἐλέγομεν πρὸς τῶν Πυθαγορείων,

[a] That is to say, $1 = 1^3$, $3 + 5 = 2^3$, $7 + 9 + 11 = 3^3$, $13 + 15 + 17 + 19 = 4^3$, $21 + 23 + 25 + 27 + 29 = 5^3$, $31 + 33 + 35 + 37 + 39 + 41 = 6^3$, and so on to infinity, the general formula being

$$\{n(n-1)+1\}+\{n(n-1)+3\}+ \ldots +\{n(n-1)+2n-1\}=n^3.$$

By putting $n=1, 2, 3 \ldots r$ in this formula and adding the results it is easily shown that

$$1^3+2^3+3^3+ \ldots +r^3=\{\tfrac{1}{2}r(r+1)\}^2,$$

a formula which was known to the Roman *agrimensores* and probably to Nicomachus. Heath (*H.G.M.* i. 109-110) shows how it was proved by the Arabian algebraist Alkarkhī in a book *Al-Fakhrī* written in the tenth or eleventh century. The proof depends on Nicomachus's theorem.

[b] Iamblichus has been considering various groups of three numbers which can be formed from the series of natural numbers, by passing over a specified number of terms, so as to become polygonal numbers. Thus $1+2+3=6$ (triangle), $1+3+5=9$ (square), $3+4+5=12$ (pentagon), $1+4+7=12$ (pentagon), $1+5+9=15$ (hexagon).

the next six in order make the sixth cube, and so on for ever.[a]

(iv.) *A Property of the Pythmen*

Iamblichus, *On Nicomachus's Introduction to Arithmetic*, ed. Pistelli 103. 10–104. 13

Since the first group,[b] starting from the unit and omitting no term, is productive of the hexad, the first group, 1, 2, 3, will be a model of those that succeed it, the groups having no common term and leaving none on one side, but 1, 2, 3 being followed by 4, 5, 6, then by 7, 8, 9, and so on in order.[c] For all these will become hexads when the unit takes the place of the decad in all cases, so reducing it to a unit. For after this manner we said 10 was called the unit of the second course [d] among the Pythagoreans, while 100

[c] In other words, Iamblichus asks us to consider any group of three consecutive numbers, the greatest of which is divisible by 3. We may represent such a group generally as $3p+1, 3p+2, 3p+3$.

[d] As Iamblichus had previously explained (*in Nicom.*, ed. Pistelli 75. 25—77. 4), the Pythagoreans looked upon a square number n^2 as a race course (δίαυλος) formed of successive numbers from 1 (as the *start*, ὕσπληξ) up to n (the *turning point*, καμπτήρ) and back again through $(n-1)$, $(n-2)$, and so on to 1 (as the *goal*, νύσσα), in this way:

$$\begin{array}{l} 1+2+3+\ .\ .\ .\ .\ +(n-1) \\ \qquad\qquad\qquad\qquad\quad + \\ \qquad\qquad\qquad\qquad\qquad n \\ \qquad\qquad\qquad\qquad\quad + \\ 1+2+3+\ .\ .\ .\ .\quad (n-1) \end{array}$$

As an example we have

$$1+2+3+\ .\ .\ .\ 10+9+8+\ .\ .\ .\ 3+2+1=10^2$$

and thence

καὶ τριωδουμέναν τὴν ἑκατοντάδα, καὶ τετρωδουμέναν τὴν χιλιάδα. ἡ μὲν γὰρ δ̄ ε̄ ϛ̄ ποιεῖ ἀριθμὸν τὸν ιε̄· ἀναγομένης δὲ τῆς δεκάδος εἰς μονάδα, ὁ πέντε προσλαβὼν αὐτὴν ἑξὰς γίνεται. πάλιν ἡ ζ̄ η̄ θ̄ συνθεῖσα ποιεῖ τὸν κδ̄ ἀριθμόν, οὗ τὰ κ̄ εἰς δύο μονάδας ἀναγαγὼν προστίθημι τῷ δ̄, καὶ ἔχω πάλιν ἑξάδα. πάλιν ῑ ιᾱ ιβ̄ συνθεὶς ποιῶ λγ̄, ὧν τὰ λ̄ τριάς ἐστιν, ἣν προσθεὶς τοῖς τρισὶν ἔχω ὁμοίως ἑξάδα, καὶ τοῦτο ὁμοίως ἔσται δι' ὅλου. καὶ ἡ μὲν πρώτη ἑξὰς οὐκ ἔχει μετάθεσιν δεκάδος εἰς μονάδα, ὡς ἂν εἰδοποιὸς καὶ στοιχεῖον τῶν μετ' αὐτὴν ὑπάρχουσα· ἡ δὲ δευτέρα μιᾶς μονάδος μετάθεσιν ἕξει, ἡ δὲ τρίτη δυεῖν καὶ ἡ τετάρτη τριῶν καὶ ἡ πέμπτη τεσσάρων καὶ ἑξῆς ἀκολούθως. ὅσαι δ' ἂν ὦσιν αἱ μετατιθέμεναι δεκάδες, τοσαῦται καὶ αἱ ἐννεάδες ἀφαιρεθήσονται ἐκ τοῦ ὅλου συστήματος, ἵνα τὸ λεῖπον ὁμοίως ἑξὰς ᾖ· τοῦ γὰρ ιε̄ μιᾶς δεκάδος ἔχοντος μετάθεσιν, ἐὰν ἀφέλω μίαν ἐννεάδα, λειφθήσεται ἑξάς. τοῦ δὲ κδ δύο ἔχοντος δεκάδας τὰς μεταποιουμένας ἐὰν ἀφέλω δύο ἐννεάδας, λειφθήσεται πάλιν ἑξάς, καὶ τοῦτο δι' ὅλου συμβήσεται.

$$10+20+30+\ \ldots\ 100+90+80+\ \ldots\ 30+20+10=10^3$$
$$100+200+300+\ \ldots\ 1000+900+800+\ \ldots\ 300+200+100 = 10^4$$

and so on. It was in virtue of these relations that the Pythagoreans spoke of 10 as the *unit of the second course* (δευτεροδουμένη μονάς), 100 as the *unit of the third course* (τριωδουμένη μονάς) and so on.

[a] The truth of Iamblichus's proposition is proved generally by Loria (*Le scienze esatte nell' antica Grecia*, pp. 841-842) in the following manner.

Let $$N = n_0 + 10n_1 + 10^2 n_2 + \ \ldots$$

was called the unit of the third course and 1000 the unit of the fourth course. Now 4, 5, 6 make the number 15. Reducing the 10 to a unit, and adding it to the 5 we get 6. Again, 7, 8, 9 when added together make the number 24, in which I reduce the 20 to two units, add them to the 4 and so again have 6. Once more, adding 10, 11, 12, I make 33, in which the 30 yields 3, and adding this to the 3 units I likewise have 6, with a similar result in all cases. The first 6 does not suffer a change of the 10 into a monad, being a kind of image and element of those that succeed it. The second has a change of one monad, the third of two, the fourth of three, the fifth of four and so on in order. The number of 10 s that have to be changed is also the number of 9 s that have to be taken away from the whole sum in order that the result may likewise be 6. In the case of 15, where there is one 10 to be changed, if I take away one 9 the remainder will be 6. In the case of 24, where there are two 10 s to be changed, if I take away two 9 s the remainder will again be 6, and this will happen in all cases.[a]

be a number written in the decimal system. Let $S(N)$ be the sum of its digits, $S^{(2)}(N)$ the sum of the digits of $S(N)$, and so on.

Now $$N - S(N) = 9(n_1 + 11n_2 + 111n_3 + \dots)$$
whence $$N \equiv S(N) \pmod{9}.$$
Similarly $$S(N) \equiv S^{(2)}N \pmod{9}$$
and so on.

Let $$S^{(k-1)}(N) \equiv S^{(k)}N \pmod{9}$$
be the last possible relation of this kind; $S^{(k)}N$ will be a number $N' \leqq 9$.

Adding all the congruences we get
$$N \equiv N' \pmod{9}, \text{ where } N' \leqq 9.$$

(*f*) Irrationality of the Square Root of 2

Aristot. *Anal. Pr.* i. 23, 41 a 26-27

Πάντες γὰρ οἱ διὰ τοῦ ἀδυνάτου περαίνοντες τὸ μὲν ψεῦδος συλλογίζονται, τὸ δ' ἐξ ἀρχῆς ἐξ ὑποθέσεως δεικνύουσιν, ὅταν ἀδύνατόν τι συμβαίνῃ τῆς ἀντιφάσεως τεθείσης, οἷον ὅτι ἀσύμμετρος ἡ διάμετρος διὰ τὸ γίνεσθαι τὰ περιττὰ ἴσα τοῖς ἀρτίοις συμμέτρου τεθείσης. τὸ μὲν οὖν ἴσα γίνεσθαι τὰ περιττὰ τοῖς ἀρτίοις συλλογίζεται, τὸ δ' ἀσύμμετρον εἶναι τὴν διάμετρον ἐξ ὑποθέσεως δείκνυσιν, ἐπεὶ ψεῦδος συμβαίνει διὰ τὴν ἀντίφασιν.

(*g*) The Theory of Proportion and Means

(i.) *Arithmetic, Geometric and Harmonic Means*

Iambl. *in Nicom. Arith. Introd.*, ed. Pistelli 100. 19-25

Μόναι δὲ τὸ παλαιὸν τρεῖς ἦσαν μεσότητες ἐπὶ Πυθαγόρου καὶ τῶν κατ' αὐτὸν μαθηματικῶν, ἀριθ-

---

Now, if N is the sum of three consecutive numbers of which the greatest is divisible by 3, we can write

$$N=(3p+1)+(3p+2)+(3p+3),$$

and the above congruence becomes

$$9p+6\equiv N'(\text{mod. } 9)$$

so that $N'\equiv 6(\text{mod. } 9)$, with the condition $N'\leqq 9$. But the only number $\leqq 9$ which is divisible by 6 is 6 itself.

Therefore $$N'=6.$$

[a] It is generally believed that the Pythagoreans were aware of the irrationality of $\sqrt{2}$ (Theodorus, for example, when proving the irrationality of numbers began with $\sqrt{3}$), and that Aristotle has indicated the method by which they proved it. The proof, interpolated in the text of Euclid as

(*f*) Irrationality of the Square Root of 2

Aristotle, *Prior Analytics* i. 23, 41 a 26-27

For all who argue *per impossibile* infer by syllogism a false conclusion, and prove the original conclusion hypothetically when something impossible follows from a contradictory assumption, as, for example, that the diagonal [of a square] is incommensurable [with the side] because odd numbers are equal to even if it is assumed to be commensurate. It is inferred by syllogism that odd numbers are equal to even, and proved hypothetically that the diagonal is incommensurate, since a false conclusion follows from the contradictory assumption.[a]

(*g*) The Theory of Proportion and Means

(i.) *Arithmetic, Geometric and Harmonic Means*

Iamblichus, *On Nicomachus's Introduction to Arithmetic*, ed. Pistelli 100. 19-25

In ancient days in the time of Pythagoras and the mathematicians of his school there were only three

x. 117 (Eucl., ed. Heiberg-Menge iii. 408-410), is roughly as follows. Suppose AC, the diagonal of a square, to be commensurable with its side AB, and let their ratio in its smallest terms be $a : b$.

Now $\quad AC^2 : AB^2 = a^2 : b^2$

and $\quad AC^2 = 2AB^2,\ a^2 = 2b^2.$

Hence a^2, and therefore a, is even.

Since $a : b$ is in its lowest terms it follows that b is odd.

Let $a = 2c$. Then $4c^2 = 2b^2$, or $b^2 = 2c^2$, so that b^2, and therefore b is even.

But b was shown to be odd, and is therefore odd and even, which is impossible. Therefore AC cannot be commensurable with AB.

ἔχει τοῦ τρίτου μέρει. γίνεται δὲ ἐν ταύτᾳ τᾷ ἀναλογίᾳ τὸ τῶν μειζόνων ὅρων διάστημα μεῖζον, τὸ δὲ τῶν μειόνων μεῖον."

(ii.) *Seven Other Means*

Nicom. *Arith. Introd.* ii. 28. 3-11, ed. Hoche 141. 4–144. 19

Τετάρτη μὲν ἡ καὶ ὑπεναντία λεγομένη διὰ τὸ ἀντικεῖσθαι καὶ ἀντιπεπονθέναι τῇ ἁρμονικῇ ὑπάρχει ὅταν ἐν τρισὶν ὅροις ὡς ὁ μέγιστος πρὸς τὸν ἐλάχιστον, οὕτως ἡ τῶν ἐλαττόνων διαφορὰ πρὸς τὴν τῶν μειζόνων ἔχῃ, οἷον

$\bar{\gamma}, \bar{\epsilon}, \bar{\varsigma},$

---

[a] *i.e.*, b is the harmonic mean between a and c if

$$\frac{a-b}{a}=\frac{b-c}{c},$$

which can be written $\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a},$

so that $\frac{1}{c}, \frac{1}{b}, \frac{1}{a}$

form an arithmetical progression, and Archytas goes on to assert that

$$\frac{a}{b}>\frac{b}{c}.$$

[b] It is easily seen how the Pythagoreans would have observed the three means in their musical studies (see A. E. Taylor, *A Commentary on Plato's Timaeus*, p. 95). They would first have noticed that when they took three vibrating strings, of which the first gave out a note an octave below the second, while the second gave out a note an octave below the third, the lengths of the strings would be proportional to 4, 2, 1. Here the διάστημα is in each case an octave. The Pythagoreans would then have noticed that if they took three

third by the same part of the third.[a] In this proportion the interval between the greater terms is the greater, that between the lesser terms is the lesser."[b]

(ii.) *Seven Other Means*

Nicomachus, *Introduction to Arithmetic* ii. 28. 3-11, ed. Hoche 141. 4–144. 19

The fourth mean, which is also called subcontrary by reason of its being reciprocal and antithetical to the harmonic, comes about when of three terms the greatest bears the same ratio to the least as the difference of the lesser terms bears to the difference of the greater,[c] as in the case of

3, 5, 6,

strings sounding a given note, its major fourth and its upper octave, the lengths of the strings would be proportional to 12, 8, 6, which are in harmonic progression. Finally they would have observed that if they took three strings sounding a note, its major fifth and its upper octave, the lengths of the strings would be proportional to 12, 9, 6, which are terms in arithmetical progression. But the fact that the means are consistently given in the order arithmetic, geometric, harmonic, and that the name "harmonic" was substituted by Archytas for the older name "subcontrary" suggests that these means had already been arithmetically defined before they were seen to be exemplified in the fundamental intervals of the octave.

[c] *i.e.*, b will be the subcontrary mean to a, c, if

$$\frac{c}{a}=\frac{b-a}{c-b}.$$

In this and the succeeding examples, following the practice of Nicomachus, it is assumed that a, b, c are in ascending order of magnitude.

ἐν γὰρ διπλασίῳ τὰ συγκριθέντα ὁρᾶται· φανερὸν δέ, καθ' ἃ ἠναντίωται τῇ ἁρμονικῇ· τῶν γὰρ αὐτῶν ἄκρων ἀμφοτέραις ὑπαρχόντων καὶ ἐν διπλασίῳ γε λόγῳ, ἐν μὲν τῇ πρὸ ταύτης ἡ τῶν μειζόνων ὑπεροχὴ πρὸς τὴν τῶν ἐλαττόνων τὸν αὐτὸν ἔσωζε λόγον, ἐν ταύτῃ δὲ ἀνάπαλιν ἡ τῶν ἐλαττόνων πρὸς τὴν τῶν μειζόνων· ἴδιον δὲ ταύτης ἰστέον ἐκεῖνο, τὸ διπλάσιον ἀποτελεῖσθαι τὸ ὑπὸ τοῦ μείζονος καὶ μέσου πρὸς τὸ ὑπὸ τοῦ μέσου καὶ ἐλαχίστου, τοῦ γὰρ πεντάκις $\bar{\gamma}$ διπλάσιον τὸ ἑξάκις $\bar{\epsilon}$.

Αἱ δὲ δύο μεσότητες πέμπτη καὶ ἕκτη παρὰ τὴν γεωμετρικὴν ἐπλάσθησαν ἀμφότεραι, διαφέρουσι δ' ἀλλήλων οὕτως· ἡ μὲν πέμπτη ἔστιν, ὅταν ἐν τρισὶν ὅροις ὡς ὁ μέσος πρὸς τὸν ἐλάχιστον οὕτω καὶ ἡ αὐτῶν τούτων διαφορὰ πρὸς τὴν τοῦ μεγίστου πρὸς τὸν μέσον, οἷον

$$\bar{\beta},\ \bar{\delta},\ \bar{\epsilon}\cdot$$

διπλάσιος γὰρ ὁ μὲν $\bar{\delta}$ τοῦ $\bar{\beta}$, μέσος ὅρος τοῦ ἐλαχίστου, ὁ δὲ $\bar{\beta}$ τοῦ $\bar{\alpha}$, ἐλαχίστων διαφορὰ πρὸς διαφορὰν μεγίστων· ὃ δ' ὑπεναντίον αὐτὴν τῇ

---

[a] An elaborate classification of ratios is given by Nicom. *Arith. Introd.* i. 17-23. They are given in a convenient form for reference by Heath, *H.G.M.* i. 101-104, with the Latin names used by Boethius in his *De Institutione Arithmetica*, which is virtually a translation of Nicomachus's work.

[b] *i.e.*, in the harmonic mean

$$\frac{c}{a}=\frac{c-b}{b-a}$$

and in the subcontrary mean

$$\frac{c}{a}=\frac{b-a}{c-b}.$$

[c] This property happens to be true of the particular

for the ratios formed are both seen to be the double.[a] It is clear in what way this mean is contrary to the harmonic; for whereas they both have the same extremes, standing in the double ratio, in the case of the former mean this was also the ratio of the difference of the greater terms towards that of the lesser, while in the case of the present mean it is the ratio of the difference of the lesser terms to that of the greater.[b] This property peculiar to the present mean deserves to be known, that the product of the greater and middle terms is double the product of the middle and least terms, for six times five is double five times three.[c]

The next two means, the fifth and sixth, were both fashioned after the geometric, and differ from each other in this way. The fifth exists when of three terms the middle bears to the least the same ratio as their difference bears to the difference between the greatest and the middle terms,[d] as in the case of

$$2, 4, 5;$$

for 4 is double 2, that is, the middle term is double the least, and 2 is double 1, that is, the difference of the least terms is double the difference of the greatest.

numbers Nicomachus has chosen, but is not in general true of the subcontrary mean. What is universally true is that if

$$\frac{c}{a}=\frac{c-b}{b-a}=r,$$

then $$abr=ab\times\frac{c}{a}=bc.$$

[d] *i.e.*, b is the fifth mean of a, c, if

$$\frac{b}{a}=\frac{b-a}{c-b}.$$

γεωμετρικῇ ποιεῖ, ἐκεῖνό ἐστιν, ὅτι ἐπὶ μὲν ἐκείνης ὡς ὁ μέσος πρὸς τὸν ἐλάττονα, οὕτως ἡ τοῦ μείζονος πρὸς τὸν μέσον ὑπεροχὴ πρὸς τὴν τοῦ μέσου πρὸς τὸν ἐλάττονα, ἐπὶ δὲ ταύτης ἀνάπαλιν ἡ τοῦ ἐλάττονος πρὸς τὴν τοῦ μείζονος· ἴδιον δ' ὅμως καὶ ταύτης ἐστὶ τὸ διπλάσιον γίνεσθαι τὸ ὑπὸ τοῦ μεγίστου καὶ μέσου τοῦ ὑπὸ τοῦ μεγίστου καὶ ἐλαχίστου, τὸ γὰρ πεντάκις $\bar{\delta}$ διπλάσιον τοῦ πεντάκις $\bar{\beta}$.

Ἡ δὲ ἕκτη γίνεται, ὅταν ἐν τρισὶν ὅροις ᾖ ὡς ὁ μέγιστος πρὸς τὸν μέσον, οὕτως ἡ τοῦ μέσου παρὰ τὸν ἐλάχιστον ὑπεροχὴ πρὸς τὴν τοῦ μεγίστου παρὰ τὸν μέσον, οἷον

$$\bar{\alpha},\ \bar{\delta},\ \bar{ϛ},$$

ἐν ἡμιολίῳ γὰρ ἑκάτεροι λόγῳ· ἐοικυῖα δ' αἰτία καὶ ταύτῃ τῆς πρὸς τὴν γεωμετρικὴν ὑπεναντιότητος, ἀναστρέφει γὰρ κἀνταῦθα ἡ τῶν λόγων ὁμοιότης ὡς ἐπὶ τῆς πέμπτης.

Καὶ αἱ μὲν παρὰ τοῖς πρόσθεν θρυλλούμεναι ἓξ μεσότητες αἵδε εἰσί, τρεῖς μὲν αἱ πρωτότυποι μέχρι

---

[a] *i.e.*, if b is the geometric mean between a and c,

$$\frac{b}{a}=\frac{c}{b}=\frac{c-b}{b-a},$$

while if b is the fifth mean between a and c,

$$\frac{b}{a}=\frac{b-a}{c-b}.$$

The property which Nicomachus notes about this mean needs generalizing as in the case of his similar remark about the fourth mean, *i.e.*, if

$$\frac{b}{a}=\frac{b-a}{c-b}=r,$$

then

$$acr=ac\times\frac{b}{a}=bc.$$

What makes it subcontrary to the geometric mean is this property, that in the case of the geometric mean the middle term bears to the lesser the same ratio as the excess of the greater term over the middle bears to that of the middle term over the lesser, while in the case of this mean a contrary relation holds. It is a peculiar property of this mean that the product of the greatest and middle terms is double the product of the greatest and least, for five times four is double of five times two.[a]

The sixth mean comes about when of three terms the greatest bears the same ratio to the middle term as the excess of the middle term over the least bears to the excess of the greatest term over the middle,[b] as in the case of

$$1, 4, 6,$$

for in each case the ratio is the sesquialter (3 : 2). No doubt, it is called subcontrary to the geometric mean because the ratios are reversed, as in the case of the fifth mean.[c]

These are then what are commonly called the six means, three prototypes which came down to Plato

[b] *i.e.*, b is the sixth mean between a and b if

$$\frac{c}{b}=\frac{b-a}{c-b}$$

[c] *i.e.*, if b is the geometric mean between a and c,

$$\frac{c}{b}=\frac{c-b}{b-a},$$

while if b is the sixth mean between a and c,

$$\frac{c}{b}=\frac{b-a}{c-b}.$$

Ἀριστοτέλους καὶ Πλάτωνος ἄνωθεν ἀπὸ Πυθαγόρου διαμεῖναι, τρεῖς δ' ἕτεραι ἐκείναις ὑπεναντίαι τοῖς μετ' ἐκείνους ὑπομνηματογράφοις τε καὶ αἱρετισταῖς ἐν χρήσει γινόμεναι· τέσσαρας δέ τινας ἑτέρας μετακινοῦντες τοὺς τούτων ὅρους τε καὶ διαφορὰς ἐπέξευρόν τινες οὐ πάνυ ἐμφανταζομένας τοῖς τῶν παλαιῶν συγγράμμασιν, ἀλλ' ὡς περιεργότερον λελεπτολογημένας, ἃς ὅμως πρὸς τὸ μὴ δοκεῖν ἀγνοεῖν ἐπιτροχαστέον τῇδέ πῃ.

Πρώτη μὲν γὰρ αὐτῶν, ἑβδόμη δὲ ἐν τῇ πασῶν συντάξει ἔστιν, ὅταν ᾖ ὡς ὁ μέγιστος πρὸς τὸν ἐλάχιστον, οὕτως καὶ ἡ τῶν αὐτῶν διαφορὰ πρὸς τὴν τῶν ἐλαττόνων, οἷον

$$\bar{\varsigma},\ \bar{\eta},\ \bar{\theta},$$

ἡμιόλιος γὰρ ὁ λόγος ἑκατέρου συγκρίσει ἐνορᾶται.

Ὀγδόη δὲ μεσότης, ἥτις τούτων δευτέρα ἐστί, γίνεται, ὅταν ὡς ὁ μέγιστος πρὸς τὸν ἐλάχιστον, οὕτως ἡ διαφορὰ τῶν ἄκρων πρὸς τὴν τῶν μειζόνων διαφοράν, οἷον

$$\bar{\varsigma},\ \bar{\zeta},\ \bar{\theta}\cdot$$

καὶ αὕτη γὰρ ἡμιολίους ἔχει τοὺς δύο λόγους.

Ἡ δὲ ἐνάτη μὲν ἐν τῇ τῶν πασῶν συντάξει, τρίτη δὲ ἐν τῷ τῶν ἐφευρημένων ἀριθμῷ ὑπάρχει, ὅταν τριῶν ὅρων ὄντων, ὃν λόγον ἔχει ὁ μέσος πρὸς

---

[a] Iamblichus says (*in Nicom.*, ed. Pistelli 101. 1-5) that the school of Eudoxus discovered these means, but in other places (*ibid.* 116. 1-4, 113. 16-18) he gives the credit, in part at least, to Archytas and Hippasus.

and Aristotle from Pythagoras, and three others subcontrary to these which came into use with later writers and partisans.[a] By playing about with the terms and their differences certain men discovered four other means which do not find a place in the writings of the ancients, but which must nevertheless be treated briefly in some fashion, although they are superfluous refinements, in order not to appear ignorant.

The first of these, or the seventh in the complete list, exists when the greatest term bears the same relation to the least as their difference bears to the difference of the lesser terms,[b] as in the case of

6, 8, 9,

for the ratio of each is seen by compounding the terms to be the sesquialter.

The eighth mean, or the second of these, comes about when the greatest term bears to the least the same ratio as the difference of the extremes bears to the difference of the greater terms,[c] as in the case of

6, 7, 9 ;

for here the two ratios are the sesquialter.

The ninth mean in the complete series, and the third in the number of those more recently discovered, comes about when there are three terms and the

[b] *i.e.*, b is the seventh mean between a and c if

$$\frac{c}{a}=\frac{c-a}{b-a}.$$

[c] *i.e.*, b is the eighth mean between a and c if

$$\frac{c}{a}=\frac{c-a}{c-b}.$$

τὸν ἐλάχιστον, τοῦτον καὶ ἡ τῶν ἄκρων ὑπεροχὴ πρὸς τὴν τῶν ἐλαχίστων ἔχῃ, ὡς

δ̅, ϛ̅, ζ̅.

Ἡ δὲ ἐπὶ πάσαις δεκάτη μὲν συλλήβδην, τετάρτη δὲ ἐν τῇ τῶν νεωτερικῶν ἐκθέσει ὁρᾶται, ὅταν ἐν τρισὶν ὅροις ᾖ ὡς ὁ μέσος πρὸς τὸν ἐλάχιστον, οὕτως καὶ ἡ διαφορὰ τῶν ἄκρων πρὸς τὴν διαφορὰν τῶν μειζόνων, οἷον

γ̅, ε̅, η̅·

ἐπιδιμερὴς γὰρ ὁ ἐν ἑκατέρᾳ συζυγίᾳ λόγος.

Ἐπὶ κεφαλαίου τοίνυν οἱ τῶν δέκα ἀναλογιῶν ὅροι ἐκκείσθωσαν ὑφ' ἓν παράδειγμα πρὸς τὸ εὐσύνοπτον,

πρώτης	α̅, β̅, γ̅,
δευτέρας	α̅, β̅, δ̅,
τρίτης	γ̅, δ̅, ϛ̅,
τετάρτης	γ̅, ε̅, ϛ̅,
πέμπτης	β̅, δ̅, ε̅,
ἕκτης	α̅, δ̅, ϛ̅,

[a] *i.e.*, b is the ninth mean between a and c if

$$\frac{b}{a}=\frac{c-a}{b-a}.$$

[b] *i.e.*, b is the tenth mean between a and c if

$$\frac{b}{a}=\frac{c-a}{c-b}.$$

[c] Pappus (iii. 18, ed. Hultsch 84. 12–86. 14) gives a similar list, but in a different order after the sixth mean. Nos. 8, 9, 10 in Nicomachus's list are respectively Nos. 9, 10, 7 in that of Pappus. Moreover Pappus omits No. 7 in the list of Nicomachus and gives as No. 8 an additional mean equivalent to the formula $\frac{c-a}{c-b}=\frac{c}{b}$. The two lists thus give five means additional to the first six.

middle bears to the least the same ratio as the difference between the extremes bears to the difference between the least terms,[a] as

4, 6, 7.

Finally, the tenth in the complete series, and the fourth in the list set out by the moderns, is seen when in three terms the middle term bears to the least the same ratio as the difference between the extremes bears to the difference of the greater terms,[b] as in the case of

3, 5, 8 ;

for the ratio in each couple is the superbipartient (5 : 3).

To sum up, then, let the terms of the ten proportions be set out in one figure so as to be taken in at a glance.[c]

		$a<b<c$
First	1, 2, 3	$\frac{b-a}{c-b}=\frac{a}{a}=\frac{b}{b}=\frac{c}{c}$; arithmetic
Second	1, 2, 4	$\frac{b-a}{c-b}=\frac{b}{c}=\frac{a}{b}$; geometric
Third	3, 4, 6	$\frac{b-a}{c-b}=\frac{a}{c}$; harmonic
Fourth	3, 5, 6	$\frac{b-a}{c-b}=\frac{c}{a}$; subcontrary to harmonic
Fifth	2, 4, 5	$\left.\begin{array}{l}\frac{b-a}{c-b}=\frac{b}{a}\\[1ex]\frac{b-a}{c-b}=\frac{c}{b}\end{array}\right\}$; subcontrary to geometric
Sixth	1, 4, 6	

ἑβδόμης	ϛ̄, η̄, θ̄,
ὀγδόης	ϛ̄, ζ̄, θ̄,
ἐνάτης	δ̄, ϛ̄, ζ̄,
δεκάτης	γ̄, ε̄, η̄.

(iii.) *Pappus's Equations between Means*

Papp. *Coll.* iii. 18. 48, ed. Hultsch 88. 5-18

Τρεῖς ἀνάλογον ἔστωσαν ὅροι οἱ Α, Β, Γ καὶ συναμφοτέρῳ μὲν τῷ Α, Γ μετὰ β̄ τῶν Β ἴσος ἐκκείσθω ὁ Δ, συναμφοτέρῳ δὲ τῷ Β, Γ ὁ Ε, τῷ δὲ Γ ὁ Ζ· λέγω ὅτι καὶ οἱ Δ, Ε, Ζ ὅροι ἀνάλογόν εἰσιν.

Ἐπεὶ γὰρ ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ, ἔσται καὶ συνθέντι ὡς συναμφότερος ὁ Α, Β πρὸς τὸν Β, οὕτως συναμφότερος ὁ Β, Γ πρὸς τὸν Γ· καὶ πάντες ἄρα οἱ ἡγούμενοι πρὸς πάντας τοὺς ἑπομένους εἰσὶν ἐν τῷ αὐτῷ λόγῳ ὡς συναμφότερος ὁ Α, Β μετὰ συναμφοτέρου τοῦ Β, Γ πρὸς συναμφότερον τὸν Β, Γ, οὕτως συναμφότερος ὁ Β, Γ πρὸς τὸν Γ. καὶ ἔστιν συναμφοτέρῳ μὲν τῷ Α, Β μετὰ συναμφοτέρου τοῦ Β, Γ ἴσος ὁ Δ, συναμφοτέρῳ δὲ

Seventh	6, 8, 9	$\dfrac{c-a}{b-a}=\dfrac{c}{a}$
Eighth	6, 7, 9	$\dfrac{c-a}{c-b}=\dfrac{c}{a}$
Ninth	4, 6, 7	$\dfrac{c-a}{b-a}=\dfrac{b}{a}$
Tenth	3, 5, 8	$\dfrac{c-a}{c-b}=\dfrac{b}{a}$ or $c=a+b$

(iii.) *Pappus's Equations between Means*

Pappus, *Collection* iii. 18. 48, ed. Hultsch 88. 5-18

Let Α, Β, Γ be three terms in [geometric] proportion [a] and let $\Delta = \mathrm{A}+\Gamma+2\mathrm{B}$, $\mathrm{E}=\mathrm{B}+\Gamma$, $\mathrm{Z}=\Gamma$; I say that Δ, Ε, Ζ are terms in [geometric] proportion.

For since $\mathrm{A}:\mathrm{B}=\mathrm{B}:\Gamma$, it follows that $\mathrm{A}+\mathrm{B}:\mathrm{B}=\mathrm{B}+\Gamma:\Gamma$; and therefore all the antecedents bear to all the consequents [b] the same ratio, so that $\mathrm{A}+\mathrm{B}+\mathrm{B}+\Gamma:\mathrm{B}+\Gamma=\mathrm{B}+\Gamma:\Gamma$. Now $\Delta = \mathrm{A}+\mathrm{B}+$

[a] According to Theon (ed. Hiller 106. 15-20), Adrastus said the geometric mean was called "both proportion *par excellence* and primary," though the other means were also commonly called proportion by some writers (*τούτων δέ φησιν ὁ Ἄδραστος μίαν τὴν γεωμετρικὴν κυρίως λέγεσθαι καὶ ἀναλογίαν καὶ πρώτην . . . κοινότερον δέ φησι καὶ τὰς ἄλλας μεσότητας ὑπ' ἐνίων καλεῖσθαι ἀναλογίας*).

[b] The expressions "antecedents," literally "leading (terms)," and "consequents," or "following (terms)," are those used in Euclid v. Def. 11 *et seq.*

τῷ Β, Γ ἴσος ὁ Ε, καὶ τῷ Γ ὁ Ζ. καὶ οἱ Δ, Ε, Ζ ἄρα ἀνάλογόν εἰσιν.

Ibid. iii. 23. 57, ed. Hultsch 102

Μεσότητες	Α Β Γ	Οἱ περιέχοντες τὰς μεσότητας τρεῖς ἐλάχιστοι ἀριθμοί
ἀριθμητική	β̄ γ̄ ᾱ ᾱ β̄ ᾱ ᾱ ᾱ	ϛ̄ δ̄ β̄
γεωμετρική	ᾱ β̄ ᾱ ᾱ ᾱ ᾱ	δ̄ β̄ ᾱ
ἁρμονική	β̄ γ̄ ᾱ β̄ ᾱ ᾱ ᾱ	ϛ̄ γ̄ β̄
ὑπεναντία	β̄ γ̄ ᾱ ᵃ β̄ β̄ ᾱ ᾱ ᾱ	ϛ̄ ε̄ β̄

[a] This is one of a series of propositions given by Pappus to the following effect. If Α, Β, Γ are three terms in geometric proportion, it is possible to form from them three other terms Δ, Ε, Ζ, being linear functions of Α, Β, Γ, which satisfy the different proportions. In this case Δ, Ε, Ζ are also in geometric proportion, but in the other examples Δ, Ε, Ζ are made to satisfy the harmonic, the subcontrary, and the fifth, sixth, eighth, ninth and tenth means of Pappus's list. The problems are, of course, problems in indeterminate analysis of the second degree. Pappus does not include solutions for the arithmetic and seventh proportions. Tannery (*Mémoires scientifiques* i., pp. 97-98) suggests as the reason that in these cases the equations of the proportions, $\Delta + \mathrm{Z} = 2\mathrm{E}$ and, $\Delta = \mathrm{E} + \mathrm{Z}$, are already linear, there is no need to assume that

$B+\Gamma$, $E=B+\Gamma$ and $Z=\Gamma$; and therefore Δ, E, Z are in [geometric] proportion.[a]

Ibid. iii. 23. 57, ed. Hultsch 102

Means	Solution in terms of A, B, Γ	The three least numbers exhibiting the means
Arithmetic	$\Delta=2A+3B+\Gamma$ $E=A+2B+\Gamma$ $Z=B+\Gamma$	6, 4, 2
Geometric	$\Delta=A+2B+\Gamma$ $E=B+\Gamma$ $Z=\Gamma$	4, 2, 1
Harmonic	$\Delta=2A+3B+\Gamma$ $E=2B+\Gamma$ $Z=B+\Gamma$	6, 3, 2
Subcontrary	$\Delta=2A+3B+\Gamma$ $E=2A+2B+\Gamma$ $Z=B+\Gamma$	6, 5, 2

$A\Gamma=B^2$, and consequently there is one indeterminate too many. But the complete results are shown in the table reproduced on these pages from Pappus (ed. Hultsch, p. 102, with explanation, pp. 100-104). The first column in the Greek table gives the means which Δ, E, Z are to satisfy. The second column gives the number of times A, B, Γ have to be taken to form Δ, E, Z respectively. In the case of the geometric progression already considered, the table shows that to form Δ we have to take A once, B twice and Γ once; to form E we have to take B once and Γ once; and to form Z we take Γ once. The third column gives the least integral values of Δ, E, Z satisfying the respective proportions (*i.e.* the values of Δ, E, Z, supposing A, B, Γ to be each unity): in the case of the geometric proportion the values are 4, 2, 1.

Μεσότητες	Α Β Γ	Οἱ περιέχοντες τὰς μεσότητας τρεῖς ἐλάχιστοι ἀριθμοί
ε′	ᾱ γ̄ ᾱ ᾱ β̄ ᾱ ᾱ ᾱ	ε δ β
ς′	ᾱ γ̄ β̄ ᾱ β̄ ᾱ ᾱ ᾱ ᾱ	ς δ α
ζ′	ᾱ ᾱ ᾱ ᾱ ᾱ ᾱ	γ β α
η′	β̄ γ̄ ᾱ ᾱ β̄ ᾱ β̄ ᾱ	ς δ γ
θ′	ᾱ β̄ ᾱ ᾱ ᾱ ᾱ ᾱ ᾱ	δ γ β
ι′	ᾱ ᾱ ᾱ ᾱ ᾱ ᾱ	γ β α

(iv.) *Plato on Means between two Squares or two Cubes*

Plat. *Tim.* 31 B–32 B

Δύο δὲ μόνω καλῶς συνίστασθαι τρίτου χωρὶς οὐ δυνατόν· δεσμὸν γὰρ ἐν μέσῳ δεῖ τινα ἀμφοῖν συναγωγὸν γίγνεσθαι. . . . εἰ μὲν οὖν ἐπίπεδον μέν, βάθος δὲ μηδὲν ἔχον ἔδει γίγνεσθαι τὸ τοῦ παντὸς σῶμα, μία μεσότης ἂν ἐξήρκει τά τε μεθ'

Means	Solution in terms of A, B, Γ	The three least numbers exhibiting the means
Fifth	$\Delta = A + 3B + \Gamma$ $E = A + 2B + \Gamma$ $Z = B + \Gamma$	5, 4, 2
Sixth	$\Delta = A + 3B + 2\Gamma$ $E = A + 2B + \Gamma$ $Z = A + B - \Gamma$	6, 4, 1
Seventh	$\Delta = A + B + \Gamma$ $E = B + \Gamma$ $Z = \Gamma$	3, 2, 1
Eighth	$\Delta = 2A + 3B + \Gamma$ $E = A + 2B + \Gamma$ $Z = 2B + \Gamma$	6, 4, 3
Ninth	$\Delta = A + 2B + \Gamma$ $E = A + B + \Gamma$ $Z = B + \Gamma$	4, 3, 2
Tenth	$\Delta = A + B + \Gamma$ $E = B + \Gamma$ $Z = \Gamma$	3, 2, 1

N.B.—For the differences between this list of means and that given by Nicomachus, see p. 122 n. *c*.

(iv.) *Plato on Means between two Squares or two Cubes*

Plato, *Timaeus* 31 B–32 B

But it is not possible that two things alone be joined without a third; for in between there must needs be some bond joining the two. . . . Now if the body of the All had had to come into being as a plane surface, having no depth, one mean would have

αὐτῆς σύνδειν καὶ ἑαυτήν, νῦν δὲ στερεοειδῆ γὰρ αὐτὸν προσῆκεν εἶναι, τὰ δὲ στερεὰ μία μὲν οὐδέποτε, δύο δὲ ἀεὶ μεσότητες συναρμόττουσιν.

(v.) *A Theorem of Archytas*

Archytas ap. Boeth. *De Inst. Mus.* iii. 11, ed. Friedlein 285-286

Demonstratio Archytae superparticularem in aequa dividi non posse.

Superparticularis proportio scindi in aequa medio proportionaliter interposito numero non potest. Id vero posterius firmiter demonstrabitur. Quam enim demonstrationem ponit Archytas, nimium fluxa est. Haec vero est huiusmodi. Sit, inquit, superparticularis proportio ·*A*·*B*·, sumo in eadem proportione minimos ·*C*·*DE*·. Quoniam igitur sunt minimi in eadem proportione ·*C*·*DE*· et sunt superparticulares, ·*DE*· numerus ·*C*· numerum parte una sua eiusque transcendit. Sit haec ·*D*·. Dico, quoniam ·*D*· non erit numerus, sed unitas. Si enim est nu-

---

[a] In other words, one mean is sufficient to connect in continuous proportion two square numbers, but two are required to connect cube numbers. Plato's remarks are equivalent to saying that

$$a^2 : ab = ab : b^2$$

and
$$a^3 : a^2b = a^2b : ab^2 = ab^2 : b^3.$$

[b] The *superparticularis ratio* (ἐπιμόριος λόγος) is the ratio in which one number contains the other and an aliquot part of it, *i.e.*, is the ratio $\frac{n+1}{n}$.

[c] That is, a geometric mean. Archytas's proof as preserved by Boethius is substantially identical with that given by Euclid in his *Sectio Canonis*, prop. 3 (*Euclid*, ed. Heiberg-

sufficed to bind together both itself and its fellow-terms; but now it is otherwise—for it behoved it to be solid in shape, and what brings solids into harmony is never one mean, but always two.[a]

(v.) *A Theorem of Archytas*

Archytas as quoted by Boethius, *On Music* iii. 11, ed. Friedlein 285-286

Archytas's proof that a superparticular ratio cannot be divided into equal parts.

A superparticular ratio[b] cannot be divided into equal parts by a mean proportional[c] placed between. That will later be more conclusively proved. For the proof which Archytas gives is very loose. It is after this manner. Let there be, he says, a superparticular ratio $A : B$.[d] I take C, $D+E$ the least numbers in the same ratio.[e] Therefore, since C, $D+E$, are the least numbers in the same ratio and are superparticulars, the number $D+E$ exceeds the number C by an aliquot part of itself and of C. Let the excess be D.[f] I say that D is not a number but a unit. For, if D is a number and an aliquot

Menge viii. 162. 7-26). It is subsequently used by Euclid (prop. 16), to show that the musical tone, whose numerical value is 9 : 8, cannot be divided into two or more equal parts.

[d] Archytas writes the smaller number first instead of second, as Euclid does.

[e] In Archytas's proof $D+E$ is represented by DE. Euclid, following his usual practice, takes a straight line divided into two parts. To find C, $D+E$, presupposes Euclid vii. 33.

[f] *i.e.*, E is supposed equal to C.

merus ·*D*· et pars est eius, qui est ·*DE*· metitur ·*D*· numerus ·*DE*· numerum; quocirca et ·*E*· numerum metietur, quo fit, ut ·*C*· quoque metiatur. Utrumque igitur ·*C*· et ·*DE*· numeros metietur ·*D*· numerus, quod est impossibile. Qui enim sunt minimi in eadem proportione quibuslibet aliis numeris, hi primi ad se invicem sunt, et solam differentiam retinent unitatem. Unitas igitur est ·*D*·. Igitur ·*DE*· numerus ·*C*· numerum unitate transcendit. Quocirca nullus incidit medius numerus, qui eam proportionem aequaliter scindat. Quo fit, ut nec inter eos, qui eandem his proportionem tenent, medius possit numerus collocari, qui eandem proportionem aequaliter scindat.

(*h*) Algebraic Equations

(i.) *Side- and Diameter-numbers*

Theon Smyr., ed. Hiller 42. 10–44. 17

Ὥσπερ δὲ τριγωνικοὺς καὶ τετραγωνικοὺς καὶ πενταγωνικοὺς καὶ κατὰ τὰ λοιπὰ σχήματα λόγους ἔχουσι δυνάμει οἱ ἀριθμοί, οὕτως καὶ πλευρικοὺς καὶ διαμετρικοὺς λόγους εὕροιμεν ἂν κατὰ τοὺς σπερματικοὺς λόγους ἐμφανιζομένους τοῖς ἀριθμοῖς. ἐκ γὰρ τούτων ῥυθμίζεται τὰ σχήματα. ὥσπερ οὖν πάντων τῶν σχημάτων κατὰ τὸν ἀνωτάτω καὶ σπερματικὸν λόγον ἡ μονὰς ἄρχει, οὕτως καὶ τῆς διαμέτρου καὶ τῆς πλευρᾶς λόγος ἐν τῇ μονάδι εὑρίσκεται. οἷον ἐκτίθενται δύο μονάδες, ὧν τὴν μὲν θῶμεν εἶναι διάμετρον, τὴν δὲ πλευράν, ἐπειδὴ

---

[a] This presupposes Euclid vii. 22.

[b] This is an inference from Euclid vii. 20. Heath (*H.G.M.*

part of $D+E$, the number D measures the number $D+E$; therefore it measures the number E, that is, the number D measures C also. The number D therefore measures both C and $D+E$, which is impossible. For the least numbers which are in the same ratio as any other numbers whatsoever are prime to one another,[a] and the only difference they retain is unity. Therefore D is a unit. Therefore the number $D+E$ exceeds the number C by a unit. Hence there is no number which is a mean between the two numbers. For this reason no mean can be placed between the numbers in the same proportion so as to divide that proportion equally.[b]

(h) Algebraic Equations

(i.) *Side- and Diameter-numbers*

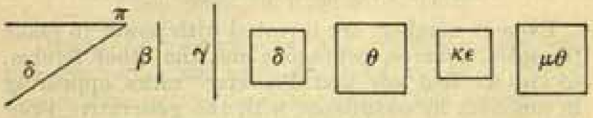
Theon of Smyrna, ed. Hiller 42. 10–44. 17

Even as numbers are invested with power to make triangles, squares, pentagons and the other figures, so also we find side and diameter[c] ratios appearing in numbers in accordance with the generative principles; for it is these which give harmony to the figures. Therefore since the unit, according to the supreme generative principle, is the starting-point of all the figures, so also in the unit will be found the ratio of the diameter to the side. To make this clear, let two units be taken, of which we set one to be a diameter and the other a side, since the unit, as the

i. 90) considers that this proposition implies the existence, at least as early as the date of Archytas (about 430–365 B.C.), of an *Elements of arithmetic* in the form which we call Euclidean.

[c] Or "diagonal."

τὴν μονάδα, πάντων οὖσαν ἀρχήν, δεῖ δυνάμει καὶ πλευρὰν εἶναι καὶ διάμετρον. καὶ προστίθεται τῇ μὲν πλευρᾷ διάμετρος, τῇ δὲ διαμέτρῳ δύο πλευραί, ἐπειδὴ ὅσον ἡ πλευρὰ δὶς δύναται, ἡ διάμετρος ἅπαξ. ἐγένετο οὖν μείζων μὲν ἡ διάμετρος, ἐλάττων δὲ ἡ πλευρά. καὶ ἐπὶ μὲν τῆς πρώτης πλευρᾶς τε καὶ διαμέτρου εἴη ἂν τὸ ἀπὸ τῆς μονάδος διαμέτρου τετράγωνον μονάδι μιᾷ ἔλαττον ἢ διπλάσιον τοῦ ἀπὸ τῆς μονάδος πλευρᾶς τετραγώνου· ἐν ἰσότητι γὰρ αἱ μονάδες· τὸ δʼ ἓν τοῦ ἑνὸς μονάδι ἔλαττον ἢ διπλάσιον. προσθῶμεν δὴ τῇ μὲν πλευρᾷ διάμετρον, τουτέστι τῇ μονάδι μονάδα· ἔσται ἡ πλευρὰ ἄρα δύο μονάδων· τῇ δὲ διαμέτρῳ προσθῶμεν δύο πλευράς, τουτέστι τῇ μονάδι δύο μονάδας· ἔσται ἡ διάμετρος μονάδων τριῶν· καὶ τὸ μὲν ἀπὸ τῆς δυάδος πλευρᾶς τετράγωνον δ̄, τὸ δʼ ἀπὸ τῆς τριάδος διαμέτρου τετράγωνον θ̄· τὸ θ̄



ἄρα μονάδι μεῖζον ἢ διπλάσιον τοῦ ἀπὸ τῆς β̄ πλευρᾶς.

Πάλιν προσθῶμεν τῇ μὲν β̄ πλευρᾷ διάμετρον τὴν τρίαδα· ἔσται ἡ πλευρὰ ε̄· τῇ δὲ τριάδι διαμέτρῳ β̄ πλευράς, τουτέστι δὶς τὰ β̄· ἔσται ζ̄· ἔσται τὸ μὲν ἀπὸ τῆς ⟨ε̄⟩ πλευρᾶς τετράγωνον κε, τὸ δὲ ἀπὸ τῆς ζ̄ ⟨διαμέτρου⟩ μθ· μονάδι ἔλασσον ἢ διπλάσιον τοῦ κε ἄρα τὸ μθ. πάλιν ἂν τῇ ⟨ε̄⟩ πλευρᾷ προσθῇς τὴν ζ̄ διάμετρον, ἔσται ιβ. κἂν τῇ ζ̄ διαμέτρῳ

beginning of all things, must have it in its capacity to be both side and diameter. Now let there be added to the side a diameter and to the diameter two sides, for as often as the square on the diameter is taken once, so often is the square on the side taken twice. The diameter will therefore become the greater and the side will become the less. Now in the case of the first side and diameter the square on the unit diameter will be less by a unit than twice the square on the unit side; for units are equal, and 1 is less by a unit than twice 1. Let us add to the side a diameter, that is, to the unit let us add a unit; therefore the [second] side will be two units. To the diameter let us now add two sides, that is, to the unit let us add two units; the [second] diameter will therefore be three units. Now the square on the side of two units will be 4, while the square on the diameter of three units will be 9; and 9 is greater by a unit than twice the square on the side 2.

Again, let us add to the side 2 the diameter 3; the [third] side will be 5. To the diameter 3 let us add two sides, that is, twice 2; the third diameter will be 7. Now the square from the side 5 will be 25, while that from the diameter 7 will be 49; and 49 is less by a unit than twice 25. Again, add to the side 5 the diameter 7; the result will be 12. And to the

προσθῇς δὶς τὴν ε̄ πλευράν, ἔσται ιζ̄. καὶ τοῦ ἀπὸ τῆς ιβ̄ τετραγώνου τὸ ἀπὸ τῆς ιζ̄ μονάδι πλέον ἢ διπλάσιον. καὶ κατὰ τὸ ἑξῆς τῆς προσθήκης ὁμοίως γιγνομένης, ἔσται τὸ ἀνάλογον ἐναλλάξ· ποτὲ μὲν μονάδι ἔλαττον, ποτὲ δὲ μονάδι πλέον ἢ διπλάσιον τὸ ἀπὸ τῆς διαμέτρου τετράγωνον τοῦ ἀπὸ τῆς πλευρᾶς· καὶ ῥηταὶ αἱ τοιαῦται καὶ πλευραὶ καὶ διάμετροι.

Procl. *in Plat. Remp.*, ed. Kroll ii. 27. 11-22

Προετίθεσαν δὲ οἱ Πυθαγόρειοι τούτου τοιόνδε

---

[a] In algebraical notation, a pair of *side-* and *diameter-numbers*, a_n, d_n are such that

$$d_n^2 - 2a_n^2 = \pm 1,$$

and the law for the formation of any pair of such numbers from the preceding pair is

$$\begin{aligned} d_n &= 2a_{n-1} + d_{n-1} \\ a_n &= a_{n-1} + d_{n-1}. \end{aligned}$$

The general proof of the property of these numbers is not given by Theon (doubtless as being well known). It can be exhibited algebraically as follows:

$$\begin{aligned} d_n^2 - 2a_n^2 &= (2a_{n-1} + d_{n-1})^2 - 2(a_{n-1} + d_{n-1})^2 \\ &= 2a_{n-1}^2 - d_{n-1}^2 \\ &= -(d_{n-1}^2 - 2a_{n-1}^2) \\ &= +(d_{n-2}^2 - 2a_{n-2}^2), \end{aligned}$$

by similar reasoning, and so on. Starting with $a_1 = 1$, $d_1 = 1$ as the first pair of side and diameter numbers, we have

$$d_1^2 - 2a_1^2 = -1$$

and therefore by the above equation we have

$$\begin{aligned} d_2^2 - 2a_2^2 &= +1, \\ d_3^2 - 2a_3^2 &= -1, \end{aligned}$$

and so on, the positive and negative signs alternating. The

diameter 7 add twice the side 5; the result will be 17. And the square of 17 is greater by a unit than twice the square of 12. Proceeding in this way in order, there will be the same alternating proportion; the square on the diameter will be now greater by a unit, now less by a unit, than twice the square on the side; and such sides and diameters are both rational.[a]

Proclus, *Commentary on Plato's Republic*, ed. Kroll ii. 27. 11-22

The Pythagoreans proposed this elegant theorem

values of the first few pairs in the series are, as Theon correctly indicates,

$$(1, 1), (2, 3), (5, 7), (12, 17),$$

the last giving, for example, the equation

$$17^2 - 2.12^2 = 289 - 288 = +1.$$

It is clear that the successive side- and diameter-numbers are rational approximations to the sides and hypotenuses of increasing isosceles right-angled triangles (hence the name), and therefore that the successive pairs give closer approximations to $\sqrt{2}$, namely

$$1, \tfrac{3}{2}, \tfrac{7}{5}, \tfrac{17}{12}, \text{ etc.},$$

and this suggests one reason why the early Greek mathematicians were so interested in them.

The series was clearly known before Plato's time, for in the famous passage about the geometrical number (*Republic* 546 c) he distinguishes between the *rational* and the *irrational* "diameter of five." In a square of side 5, the diagonal or diameter is $\sqrt{50}$, and this is the "irrational diameter of five"; the "rational diameter" was the integral approximation $\sqrt{50 - 1} = 7$, which we have seen above to be the third diameter number.

In fact, since the publication of Kroll's edition of Proclus's commentary, the belief that these approximations are Pythagorean has been fully confirmed, as the next passage will show.

θεώρημα γλαφυρὸν περὶ τῶν διαμέτρων καὶ πλευρῶν, ὅτι ἡ μὲν διάμετρος προσλαβοῦσα τὴν πλευράν, ἧς ἐστιν διάμετρος, γίνεται πλευρά, ἡ δὲ πλευρὰ ἑαυτῇ συντεθεῖσα καὶ προσλαβοῦσα τὴν διάμετρον τὴν ἑαυτῆς γίνεται διάμετρος. καὶ τοῦτο δείκνυται διὰ τῶν ἐν τῷ δευτέρῳ Στοιχείων γραμμικῶς ἀπ' ἐκείνου. ἐὰν εὐθεῖα τμηθῇ δίχα, προσλάβῃ δὲ εὐθεῖαν, τὸ ἀπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τὸ ἀπὸ ταύτης μόνης τετράγωνα διπλάσια τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς συγκειμένης ἐκ τῆς ἡμισείας καὶ τῆς προσληφθείσης.

(ii.) *The "Bloom" of Thymaridas*

Iambl. *in Nicom. Arith. Introd.*, ed. Pistelli 62. 18–63. 2

Ἐντεῦθεν καὶ ἡ ἔφοδος τοῦ Θυμαριδείου ἐπ-

---

[a] This is Euclid ii. 10, which asserts that if ΑΓ is bisected at Β

Α Β Γ Δ

and produced to Δ, then

$$A\Delta^2 + \Delta\Gamma^2 = 2AB^2 + 2B\Delta^2.$$

If $AB = x$, $\Gamma\Delta = y$, this gives

$$(2x+y)^2 + y^2 = 2x^2 + 2(x+y)^2$$

or

$$(2x+y)^2 - 2(x+y)^2 = 2x^2 - y^2.$$

Therefore, if (x, y) are a pair of numbers satisfying one of the equations $2x^2 - y^2 = \pm 1$, then $(x+y)$, $(2x+y)$ are another pair of numbers satisfying the other equation.

Proclus is not quoting exactly the Euclidean enunciation, for which see Euclid, ed. Heiberg-Menge i. 146. 15-22.

[b] Thymaridas was apparently an early Pythagorean, not

about the diameters and sides, that when the diameter receives the side of which it is diameter it becomes a side, while the side, added to itself and receiving its diameter, becomes a diameter. And this is proved graphically in the second book of the *Elements* by him [*sc.* Euclid]. If a straight line be bisected and a straight line be added to it, the square on the whole line including the added straight line and the square on the latter by itself are together double of the square on the half and of the square on the straight line made up of the half and the added straight line.[a]

(ii.) *The "Bloom" of Thymaridas*[b]

Iamblichus, *On Nicomachus's Introduction to Arithmetic*, ed. Pistelli 62. 18–63. 2

The method of the "bloom" of Thymaridas was

later than the time of Plato, who lived at Paros. The name ἐπάνθημα (*flower* or *bloom*) given to his method shows that it must have been widely known in antiquity, though the term is not confined to this particular proposition. It is presumably used to give a sense of distinction, much as we say "flower of the army." The Greek is unfortunately most obscure, but the meaning was successfully extracted by Nesselman (*Die Algebra der Griechen*, pp. 232-236), who is followed by Gow (*History of Greek Mathematics*, p. 97), Cantor (*Vorlesungen* i³. 158-159), Loria (*Le scienze esatte nell' antica Grecia*, pp. 807-809), and Heath (*H.G.M.*, i. 94-96, *Diophantus of Alexandria*, 2nd ed., pp. 114-116). The "bloom" is a rule for solving n simultaneous equations connecting n unknown quantities, and states in effect:

(1) if $\quad x+x_1+x_2=\mathrm{S}$,

while $\quad x+x_1=s_1,\ x+x_2=s_2$,

then $\quad x=s_1+s_2-\mathrm{S}$;

ανθήματος ἐλήφθη. ὡρισμένων γὰρ ἢ ἀορίστων μερισαμένων ὡρισμένον τι καὶ ἑνὸς οὑτινοσοῦν τοῖς λοιποῖς καθ' ἕκαστον συντεθέντος, τὸ ἐκ πάντων ἀθροισθὲν πλῆθος ἐπὶ μὲν τριῶν μετὰ τὴν ἐξ ἀρχῆς ὁρισθεῖσαν ποσότητα ὅλον τῷ συγκριθέντι προσνέμει τ' ἀφ' οὗ τὸ λεῖπον καθ' ἕκαστον τῶν λοιπῶν ἀφαιρεθήσεται, ἐπὶ δὲ τεσσάρων τὸ ἥμισυ καὶ ἐπὶ πέντε τὸ τρίτον καὶ ἐπὶ ἓξ τὸ τέταρτον καὶ ἀεὶ ἀκολούθως, δυάδος κἀνταῦθα διαφορᾶς ἐπιφαινομένης πρός τε τὴν ποσότητα τῶν μεριζομένων καὶ πρὸς τὴν τοῦ μορίου κλῆσιν.

---

(2) if $x+x_1+x_2+x_3=S$,
while $x+x_1=s_1,\ x+x_2=s_2,\ x+x_3=s_3$,
then $x=\dfrac{s_1+s_2+s_3-S}{2}$.

(3) while generally, if $x+x_1+x_2+\ .\ .\ +x_{n-1}=S$,
while $x+x_1=s_1,\ x+x_2=s_2\ .\ .\ .\ x+x_{n-1}=s_{n-1}$,
then $x=\dfrac{s_1+s_2+\ .\ .\ .\ +s_{n-1}-S}{n-2}$.

Iamblichus goes on to show how other equations can be

thence taken.[a] When any determined or undefined quantities amount to a given sum, and the sum of one of them *plus* every other [in pairs] is given, the sum of these pairs minus the first given sum is, if there be three quantities, equal to the quantity which was added to all the rest [in the pairs]; if there be four quantities, one-half is so equal; if there be five quantities, one-third; if there be six quantities, one-fourth, and so on continually, there being always a difference of 2 between the number of quantities to be divided and the denomination of the part.

reduced to this form, so that the rule "does not leave us in the lurch" (*οὐ παρέλκει*) in these cases.

One of the most interesting features in this passage is the distinction between the *ὡρισμένον*, or known quantity, and the *ἀόριστον*, or unknown. This anticipates the phrase *πλῆθος μονάδων ἀόριστον*, "an undefined number of units," by which Diophantus was later to describe his unknown quantity. Indeed, Thymaridas was already bordering on that indeterminate analysis which Diophantus was so brilliantly to develop; he has passed beyond the realm of strict arithmetic.

[a] This passage immediately follows the section describing how gnomons of polygonal numbers are formed; see pp. 86-89 n. *a*, where it is shown that if n is the number of sides in the polygon, the successive gnomonic numbers differ by $n-2$.

IV. PROCLUS'S SUMMARY

IV. PROCLUS'S SUMMARY

Procl. *in Eucl.* i., ed. Friedlein 64. 16–70. 18

Ἐπεὶ δὲ χρὴ τὰς ἀρχὰς καὶ τῶν τεχνῶν καὶ τῶν ἐπιστημῶν πρὸς τὴν παροῦσαν περίοδον σκοπεῖν, λέγομεν, ὅτι παρ' Αἰγυπτίοις μὲν εὑρῆσθαι πρῶτον ἡ γεωμετρία παρὰ τῶν πολλῶν ἱστόρηται, ἐκ τῆς τῶν χωρίων ἀναμετρήσεως λαβοῦσα τὴν γένεσιν. ἀναγκαία γὰρ ἦν ἐκείνοις αὕτη διὰ τὴν ἄνοδον τοῦ Νείλου τοὺς προσήκοντας ὅρους ἑκάστοις ἀφανί-

[a] The course of Greek geometry from the earliest days to the time of Euclid is reviewed in the few pages from Proclus's *Commentary on Euclid*, Book i., which are here reproduced. This "Summary" of Proclus has often been called the "Eudemian summary," on the assumption that it is extracted from the lost *History of Geometry* by Eudemus, the pupil of Aristotle. But the latter part dealing with Euclid cannot have been written by Eudemus, who preceded Euclid, nor is there any stylistic reason for attributing the earlier and later portions to different hands. Heath (*The Thirteen Books of Euclid's Elements*, i., pp. 37, 38, and *H.G.M.* i. 119, 120) gives arguments for believing that the author cannot have been Proclus himself, and suggests that the body of the summary was taken by Proclus from a compendium by some writer later than Eudemus, though the earlier portion was based, directly or indirectly, on Eudemus's *History*. The summary was written primarily for an understanding of the way in which the elements of geometry had come into being. The more advanced discoveries are therefore omitted or mentioned only in passing. Proclus himself lived from A.D. 410 to 485. On the death of Syrianus he became head of the

IV. PROCLUS'S SUMMARY [a]

Proclus, *On Euclid* i., ed. Friedlein 64. 16–70. 18

Since it behoves us to examine the beginnings both of the arts and of the sciences with reference to the present cycle [of the universe], we say that according to most accounts geometry was first discovered among the Egyptians,[b] taking its origin from the measurement of areas. For they found it necessary by reason of the rising of the Nile, which wiped out

Neo-Platonic school at Athens, and his *Commentary on Euclid*, Book i., seems to be a revised edition of his lectures to beginners in mathematics (Heath, *The Thirteen Books of Euclid's Elements*, i., p. 31). This commentary is one of the two main sources for the history of Greek geometry, the other being the *Collection* of Pappus.

[b] The Egyptian origin of geometry is taught by Herodotus, ii. 109, where it is asserted that Sesostris (Ramses II, *c.* 1300 B.C.) divided the land among the Egyptians in equal rectangular plots, on which an annual tax was levied; when therefore the river swept away a portion of a plot, the owner applied for a reduction of tax, and surveyors had to be sent down to report. In this he saw the origin of geometry, and this story may be the source of Proclus's account, as also of the similar accounts in Heron, *Geometrica* 2, ed. Heiberg 176. 1-13, Diodorus Siculus i. 69, 81 and Strabo xvii. c. 3. Aristotle also finds the origin of mathematics among the Egyptians, but in the existence of a leisured class of priests, not in a practical need (*Metaphysica* A 1, 981 b 23). The subject is fully dealt with in *H.G.M.* i. 121, 122, and an account of Egyptian geometry is given in succeeding pages.

ζοντος. καὶ θαυμαστὸν οὐδὲν ἀπὸ τῆς χρείας ἄρξασθαι τὴν εὕρεσιν καὶ ταύτης καὶ τῶν ἄλλων ἐπιστημῶν, ἐπειδὴ πᾶν τὸ ἐν γενέσει φερόμενον ἀπὸ τοῦ ἀτελοῦς εἰς τὸ τέλειον πρόεισιν. ἀπὸ αἰσθήσεως οὖν εἰς λογισμὸν καὶ ἀπὸ τούτου ἐπὶ νοῦν ἡ μετάβασις γένοιτο ἂν εἰκότως. ὥσπερ οὖν παρὰ τοῖς Φοίνιξιν διὰ τὰς ἐμπορείας καὶ τὰ συναλλάγματα τὴν ἀρχὴν ἔλαβεν ἡ τῶν ἀριθμῶν ἀκριβὴς γνῶσις, οὕτω δὴ καὶ παρ' Αἰγυπτίοις ἡ γεωμετρία διὰ τὴν εἰρημένην αἰτίαν εὕρηται.

Θαλῆς δὲ πρῶτον εἰς Αἴγυπτον ἐλθὼν μετήγαγεν εἰς τὴν Ἑλλάδα τὴν θεωρίαν ταύτην καὶ πολλὰ μὲν αὐτὸς εὗρεν, πολλῶν δὲ τὰς ἀρχὰς τοῖς μετ' αὐτὸν ὑφηγήσατο, τοῖς μὲν καθολικώτερον ἐπιβάλλων, τοῖς δὲ αἰσθητικώτερον. μετὰ δὲ τοῦτον Ἀμέριστος[1] ὁ Στησιχόρου τοῦ ποιητοῦ ἀδελφός, ὃς ἐφαψάμενος τῆς περὶ γεωμετρίαν σπουδῆς μνη-

[1] Μάμερκος Friedlein, following a correction in the oldest MS.

---

[a] Thales (c. 624–547 B.C.), one of the "Seven Wise Men" of ancient Greece, is universally acknowledged as the founder of Greek geometry, astronomy and philosophy. His greatest fame in antiquity rested on his prediction of the total eclipse of the sun of May 28, 585 B.C., which led to the cessation of hostilities between the Medes and Lydians and a lasting

everybody's proper boundaries. Nor is there anything surprising in that the discovery both of this and of the other sciences should have its origin in a practical need, since everything which is in process of becoming progresses from the imperfect to the perfect. Thus the transition from perception to reasoning and from reasoning to understanding is natural. Just as exact knowledge of numbers received its origin among the Phoenicians by reason of trade and contracts, even so geometry was discovered among the Egyptians for the aforesaid reason.

Thales[a] was the first to go to Egypt and bring back to Greece this study; he himself discovered many propositions, and disclosed the underlying principles of many others to his successors, in some cases his method being more general, in others more empirical. After him Ameristus,[b] the brother of the poet Stesichorus, is mentioned as having touched the study

peace (Herodotus i. 74): what Thales probably did was to predict the *year* in which the eclipse would take place, an achievement by no means beyond the astronomical powers of the age. Thales was noted for his political sense. He urged the separate states of Ionia, threatened by the encroachment of the Lydians, to form a federation with a capital at Teos; and his successful dissuasion of his fellow-Milesians from accepting the overtures of Croesus, king of the Lydians, may have had an influence on the favourable terms later granted to Miletus by Cyrus, king of the Persians, though the main reason for this preferential treatment was probably commercial. In philosophy Thales taught that the all is water. For his mathematical discoveries, see *infra*, pp. 164-169.

[b] The name is uncertain. Friedlein, in suggesting Mamercus, observes that Suidas gives a brother of Stesichorus as Mamertinus, which could easily arise out of Mamercus. Another reading is Mamertius. Nothing more is known about him. Stesichorus, the lyric poet, flourished *c.* 611 B.C.

μονεύεται, καὶ Ἱππίας ὁ Ἠλεῖος ἱστόρησεν ὡς ἐπὶ γεωμετρίᾳ δόξαν αὐτοῦ λαβόντος. ἐπὶ δὲ τούτοις Πυθαγόρας τὴν περὶ αὐτὴν φιλοσοφίαν εἰς σχῆμα παιδείας ἐλευθέρου μετέστησεν, ἄνωθεν τὰς ἀρχὰς αὐτῆς ἐπισκοπούμενος καὶ ἀύλως καὶ νοερῶς τὰ θεωρήματα διερευνώμενος, ὃς δὴ καὶ τὴν τῶν ἀνὰ λόγον[1] πραγματείαν καὶ τὴν τῶν κοσμικῶν σχημάτων σύστασιν ἀνεῦρεν. μετὰ δὲ τοῦτον Ἀναξαγόρας ὁ Κλαζομένιος πολλῶν ἐφήψατο τῶν κατὰ γεωμετρίαν καὶ Οἰνοπίδης ὁ Χῖος, ὀλίγῳ νεώτερος ὢν Ἀναξαγόρου, ὧν καὶ ὁ Πλάτων ἐν τοῖς ἀντερασταῖς ἐμνημόνευσεν ὡς ἐπὶ τοῖς μαθήμασι δόξαν λαβόντων.

[1] τῶν ἀνὰ λόγον coni. Diels; τῶν ἀλόγων Friedlein.

---

[a] The well-known Sophist, born about 460 B.C., whose various accomplishments are described in Plato's *Hippias Minor*. He claimed to have gone once to the Olympic Games with everything that he wore made by himself, as well as all kinds of works in prose and verse of his own composition. His system of mnemonics enabled him to remember any string of fifty names which he had heard once. The unmathematical Spartans, however, could not appreciate his genius, and from them he could get no fees. His chief mathematical discovery was the curve known as the quadratrix, which could be used for trisecting an angle or squaring the circle (see *infra*, pp. 336-347).

[b] The life of Pythagoras is shrouded in mystery. He was probably born in Samos about 582 B.C. and migrated about 529 B.C. to Crotona, the Dorian colony in southern Italy, where a semi-religious brotherhood sprang up round him. This brotherhood was subjected to severe persecution in the fifth century B.C., and the Pythagoreans then took their

of geometry, and Hippias of Elis [a] spoke of him as having acquired a reputation for geometry. After these Pythagoras [b] transformed this study into the form of a liberal education, examining its principles from the beginning and tracking down the theorems immaterially and intellectually; he it was who discovered the theory of proportionals [c] and the construction of the cosmic figures. After him Anaxagoras of Clazomenae [d] touched many questions affecting geometry, and so did Oenopides of Chios,[e] being a little younger than Anaxagoras, both of whom Plato mentioned in the *Rivals* [f] as having acquired a reputation for mathematics.

doctrines into Greece proper. Apart from important mathematical discoveries, noticed in a separate chapter, the Pythagoreans discovered the numerical ratios of the notes in the octave, and in astronomy conceived of the earth as a globe moving with the other planets about a central luminary.

[c] Friedlein's reading is τῶν ἀλόγων, "irrationals," but there is grave difficulty in believing that Pythagoras could have developed a theory of irrationals; in fact, a Pythagorean is said to have been drowned at sea for his impiety in disclosing the existence of irrationals. There is an alternative reading τῶν ἀναλόγων, and the true reading could easily be τῶν ἀναλογιῶν, or τῶν ἀνὰ λόγον, "proportionals."

[d] c. 500–428 B.C. Clazomenae was a town near Smyrna. All we know about the mathematics of Anaxagoras is that he wrote on the squaring of the circle while in prison (*infra*, p. 308) and may have written a book on perspective (Vitruvius, *De architectura* vii. praef. 11).

[e] Oenopides was primarily an astronomer, and Eudemus is believed to have credited him with the discovery of the obliquity of the ecliptic and the period of the Great Year (Theon of Smyrna, ed. Hiller 198. 14–16). In mathematics Proclus attributed to him the discovery of Eucl. i. 12 and i. 23.

[f] Plat. *Erastae* 132 A, B. Socrates finds two lads in the school of Dionysius disputing about Anaxagoras or Oenopides; they seemed to be drawing circles and indicating certain inclinations by placing their hands at an angle.

Ἐφ' οἷς Ἱπποκράτης ὁ Χῖος ὁ τὸν τοῦ μηνίσκου τετραγωνισμὸν εὑρών, καὶ Θεόδωρος ὁ Κυρηναῖος ἐγένοντο περὶ γεωμετρίαν ἐπιφανεῖς. πρῶτος γὰρ ὁ Ἱπποκράτης τῶν μνημονευομένων καὶ στοιχεῖα συνέγραψεν. Πλάτων δ' ἐπὶ τούτοις γενόμενος μεγίστην ἐποίησεν ἐπίδοσιν τά τε ἄλλα μαθήματα καὶ τὴν γεωμετρίαν λαβεῖν διὰ τὴν περὶ αὐτὰ σπουδήν, ὅς που δῆλός ἐστι καὶ τὰ συγγράμματα τοῖς μαθηματικοῖς λόγοις καταπυκνώσας καὶ πανταχοῦ τὸ περὶ αὐτὰ θαῦμα τῶν φιλοσοφίας ἀντεχομένων ἐπεγείρων. ἐν δὲ τούτῳ τῷ χρόνῳ καὶ Λεωδάμας ὁ Θάσιος ἦν καὶ Ἀρχύτας ὁ Ταραντῖνος καὶ Θεαίτητος ὁ Ἀθηναῖος, παρ' ὧν ἐπηυξήθη τὰ θεωρήματα καὶ προῆλθεν εἰς ἐπιστημονικωτέραν σύστασιν.

Λεωδάμαντος δὲ νεώτερος ὁ Νεοκλείδης καὶ ὁ τούτου μαθητὴς Λέων, οἳ πολλὰ προσευπόρησαν τοῖς πρὸ αὐτῶν, ὥστε τὸν Λέοντα καὶ τὰ στοιχεῖα συνθεῖναι τῷ τε πλήθει καὶ τῇ χρείᾳ τῶν δεικνυμένων ἐπιμελέστερον, καὶ διορισμοὺς εὑρεῖν, πότε δυνατόν ἐστι τὸ ζητούμενον πρόβλημα καὶ πότε ἀδύνατον. Εὔδοξος δὲ ὁ Κνίδιος, Λέοντος μὲν ὀλίγῳ νεώτερος, ἑταῖρος δὲ τῶν περὶ Πλάτωνα

---

[a] Hippocrates was in Athens from about 450 to 430 B.C. For his mathematical achievements, see *infra*, pp. 234-253.

[b] Our chief knowledge of Theodorus comes from the *Theaetetus* of Plato, whose mathematical teacher he is said to have been (Diog. Laert. ii. 103); see *infra*, pp. 380-383.

[c] Proclus (*in Eucl.* i., ed. Friedlein 72 *et seq.*) explains that the *elements* in geometry are leading theorems having to those which follow the relation of an all-pervading principle; he compares them with the letters of the alphabet in relation

After them Hippocrates of Chios,[a] who discovered the quadrature of the lune, and Theodorus of Cyrene[b] became distinguished in geometry. For Hippocrates is the first of those mentioned as having compiled *elements*.[c] Plato,[d] who came after them, made the other branches of mathematics as well as geometry take a very great step forward by his zeal for them; and it is obvious how he filled his writings with mathematical arguments and everywhere stirred up admiration for mathematics in those who took up philosophy. At this time also lived Leodamas of Thasos[e] and Archytas of Taras[f] and Theaetetus of Athens,[g] by whom the theorems were increased and an advance was made towards a more scientific grouping.

Younger than Leodamas were Neoclides and his pupil Leon, who added many things to those known before them, so that Leon was able to make a collection of the *elements* in which he was more careful in respect both of the number and of the utility of the things proved; he also discovered *diorismi*, showing when the problem investigated can be solved and when not.[h] Eudoxus of Cnidos, a little younger than Leon and an associate of Plato's school, was the first

to language; and they have, indeed, the same name in Greek.

[d] See *infra*, pp. 386-405.

[e] All we know about him is that Plato is said to have explained or communicated to him the method of analysis (Diog. Laert. iii. 24, Procl. *in Eucl.* i., ed. Friedlein 211. 19-23).

[f] For Archytas, see *supra*, p. 4 n. *a*.

[g] See *infra*, pp. 378-383.

[h] We have no further knowledge of Neoclides and Leon. A good example of a *diorismos* is given in Plato, *Meno* 86 E—87 B (*infra*, pp. 394-397), which incidentally shows that Leon was not the first in this field.

γενόμενος, πρῶτος τῶν καθόλου καλουμένων θεωρημάτων τὸ πλῆθος ηὔξησεν καὶ ταῖς τρισὶν ἀναλογίαις ἄλλας τρεῖς προσέθηκεν καὶ τὰ περὶ τὴν τομὴν ἀρχὴν λαβόντα παρὰ Πλάτωνος εἰς πλῆθος προήγαγεν καὶ ταῖς ἀναλύσεσιν ἐπ᾿ αὐτῶν χρησάμενος. Ἀμύκλας δὲ ὁ Ἡρακλεώτης, εἷς τῶν Πλάτωνος ἑταίρων καὶ Μέναιχμος ἀκροατὴς ὢν Εὐδόξου καὶ Πλάτωνι δὲ συγγεγονὼς καὶ ὁ ἀδελφὸς αὐτοῦ Δεινόστρατος ἔτι τελεωτέραν ἐποίησαν τὴν ὅλην γεωμετρίαν. Θεύδιος δὲ ὁ Μάγνης ἔν τε τοῖς μαθήμασιν ἔδοξεν εἶναι διαφέρων καὶ κατὰ τὴν ἄλλην φιλοσοφίαν· καὶ γὰρ τὰ στοιχεῖα καλῶς συνέταξεν καὶ πολλὰ τῶν μερικῶν[1] καθολικώτερα ἐποίησεν. καὶ μέντοι καὶ ὁ Κυζικηνὸς Ἀθήναιος κατὰ τοὺς αὐτοὺς γεγονὼς χρόνους καὶ ἐν τοῖς ἄλλοις μὲν μαθήμασι, μάλιστα δὲ κατὰ γεωμετρίαν ἐπιφανὴς ἐγένετο. διῆγον οὖν οὗτοι μετ᾿ ἀλλήλων ἐν Ἀκαδημίᾳ κοινὰς ποιούμενοι τὰς ζητήσεις. Ἑρμότιμος δὲ ὁ Κολοφώνιος τὰ ὑπ᾿ Εὐδόξου προηυπορημένα καὶ Θεαιτήτου προήγαγεν ἐπὶ πλέον

[1] ὁρικῶν Friedlein.

---

[a] For Eudoxus, one of the great mathematicians of all time, see *infra*, pp. 408-415. He lived *c.* 408–355 B.C. What the "so-called general theorems" may be is uncertain; Heath (*H.G.M.* i. 323) suggests theorems which are "true of everything falling under the conception of magnitude, as are the definitions and theorems forming part of Eudoxus's own theory of proportion." The three means which Eudoxus is said to have added to those already known are the three sub-contrary means (*supra*, pp. 114-121). Iamblichus (*in Nicom.*, 101. 1-5) also attributes them to Eudoxus, but in other places (113. 16-18, 116. 1-4) he assigns them to Archytas and Hippasus. It is disputed whether the "section" to which Eudoxus devoted his attention means sections of solids

to increase the number of the so-called general theorems; to the three proportions he added another three, and increased the number of theorems about the section, which had their origin with Plato, applying the method of analysis to them.[a] Amyclas of Heraclea,[b] one of the friends of Plato, and Menaechmus,[c] a pupil of Eudoxus who had associated with Plato, and his brother Dinostratus[d] made the whole of geometry still more perfect. Theudius[e] of Magnesia seemed to excel both in mathematics and in the rest of philosophy; for he made an admirable arrangement of *elements* and made many particular propositions more general. Again, Athenaeus[e] of Cyzicus, who lived about those times, became famous in other branches of mathematics but mostly in geometry. They spent their time together in the Academy, conducting their investigations in common. Hermotimus[e] of Colophon advanced farther the investigations begun by Eudoxus and Theaetetus; he

by planes, which was the older view and that favoured by Tannery (*La géométrie grecque*, p. 76), or the "golden section" (division of a line in extreme and mean ratio, Eucl. ii. 11), a view put forward by Bretschneider in 1870 (*Die Geometrie und die Geometer vor Eukleides*, pp. 167-169). For discussions of this interesting question see Loria, *Le scienze esatte nell' antica Grecia*, pp. 139-142, Heath, *H.G.M.* i. 324-325.

[b] The correct spelling appears to be Amyntas, though Diogenes Laertius (iii. 46) speaks of Amyclas of Heraclea as a pupil of Plato and in another place (ix. 40) says that a certain Pythagorean Amyclas dissuaded Plato from burning the works of Democritus. Heraclea was in Pontus.

[c] He discovered the conic sections, see *infra*, p. 283 n. *a*.

[d] He applied the quadratrix (probably discovered by Hippias) to the squaring of the circle.

[e] No more is known of Theudius, Athenaeus or Hermotimus.

καὶ τῶν στοιχείων πολλὰ ἀνεῦρε καὶ τῶν τόπων τινὰ συνέγραψεν. Φίλιππος δὲ ὁ Μεδμαῖος,[1] Πλάτωνος ὢν μαθητὴς καὶ ὑπ' ἐκείνου προτραπεὶς εἰς τὰ μαθήματα, καὶ τὰς ζητήσεις ἐποιεῖτο κατὰ τὰς Πλάτωνος ὑφηγήσεις καὶ ταῦτα προύβαλλεν ἑαυτῷ, ὅσα ᾤετο τῇ Πλάτωνος φιλοσοφίᾳ συντελεῖν.

Οἱ μὲν οὖν τὰς ἱστορίας ἀναγράψαντες μέχρι τούτου προάγουσι τὴν τῆς ἐπιστήμης ταύτης τελείωσιν. οὐ πολὺ δὲ τούτων νεώτερός ἐστιν Εὐκλείδης ὁ τὰ στοιχεῖα συναγαγὼν καὶ πολλὰ μὲν τῶν Εὐδόξου συντάξας, πολλὰ δὲ τῶν Θεαιτήτου τελεωσάμενος, ἔτι δὲ τὰ μαλακώτερον δεικνύμενα τοῖς ἔμπροσθεν εἰς ἀνελέγκτους ἀποδείξεις ἀναγαγών. γέγονε δὲ οὗτος ὁ ἀνὴρ ἐπὶ τοῦ πρώτου Πτολεμαίου· καὶ γὰρ ὁ Ἀρχιμήδης ἐπιβαλὼν καὶ τῷ πρώτῳ μνημονεύει τοῦ Εὐκλείδου, καὶ μέντοι καί φασιν ὅτι Πτολεμαῖος ἤρετό ποτε αὐτόν, εἴ τίς ἐστιν περὶ γεωμετρίαν ὁδὸς συντομωτέρα τῆς στοιχειώσεως· ὁ δὲ ἀπεκρίνατο, μὴ εἶναι βασιλικὴν ἀτραπὸν ἐπὶ γεωμετρίαν· νεώτερος μὲν οὖν ἐστι τῶν περὶ Πλάτωνα, πρεσβύτερος δὲ Ἐρατοσθένους καὶ

[1] Μενδαῖος Friedlein.

[a] Almost certainly the same as Philippus of Opus, who is said to have revised and published the *Laws* of Plato and (wrongly) to have written the *Epinomis*. Suidas notes a number of astronomical and mathematical works by him.

[b] Not much more is known about the life of Euclid than is contained in this passage (see Heath, *The Thirteen Books of Euclid's Elements*, vol. i., pp. 1-6 and *H.G.M.* i. 354-357). The summary of Euclid's achievement in the *Elements* is a very fair one, agreeing with the considered judgement of Heath (*H.G.M.* i. 217): "There is therefore probably little

discovered many propositions in the *elements* and compiled some portion of the theory of loci. Philippus of Medma,[a] a disciple of Plato and by him diverted to mathematics, not only made his investigations according to Plato's directions but set himself to do such things as he thought would fit in with the philosophy of Plato.

Those who have compiled histories carry the development of this science up to this point. Not much younger than these is Euclid, who put together the *elements*, arranging in order many of Eudoxus's theorems, perfecting many of Theaetetus's, and also bringing to irrefutable demonstration the things which had been only loosely proved by his predecessors. This man lived in the time of the first Ptolemy; for Archimedes, who came immediately after the first Ptolemy, makes mention of Euclid; and further they say that Ptolemy once asked him if there was in geometry a way shorter than that of the elements; he replied that there was no royal road to geometry.[b] He is therefore younger than the pupils of Plato, but

in the whole compass of the *Elements* of Euclid, except the new theory of proportion due to Eudoxus and its consequences, which was not in substance included in the recognized content of geometry and arithmetic by Plato's time, although the form and arrangement of the subject-matter and the method employed in particular cases were different from what we find in Euclid" (*cf. H.G.M.* i. 357). As Plato died in 347 B.C., and Archimedes was born in 287 B.C., Euclid must have flourished about 300 B.C.; Ptolemy I reigned from 306 to 283 B.C. Had not the confusion been common in the Middle Ages, it would scarcely be necessary to point out that this Euclid is to be distinguished from Euclid of Megara, the philosopher, who lived about 400 B.C.

[b] A story about there being no royal road to geometry is also told of Menaechmus and Alexander (Stobaeus, *Ecl.* ii. 31, ed. Wachsmuth 115).

Ἀρχιμήδους. οὗτοι γὰρ σύγχρονοι ἀλλήλοις, ὥς πού φησιν Ἐρατοσθένης. καὶ τῇ προαιρέσει δὲ Πλατωνικός ἐστι καὶ τῇ φιλοσοφίᾳ ταύτῃ οἰκεῖος, ὅθεν δὴ καὶ τῆς συμπάσης Στοιχειώσεως τέλος προεστήσατο τὴν τῶν καλουμένων Πλατωνικῶν σχημάτων σύστασιν. πολλὰ μὲν οὖν καὶ ἄλλα τοῦ ἀνδρὸς τούτου μαθηματικὰ συγγράμματα θαυμαστῆς ἀκριβείας καὶ ἐπιστημονικῆς θεωρίας μεστά. τοιαῦτα γὰρ καὶ τὰ Ὀπτικὰ καὶ τὰ Κατοπτρικά, τοιαῦται δὲ καὶ αἱ κατὰ μουσικὴν στοιχειώσεις, ἔτι δὲ τὸ Περὶ διαιρέσεων βιβλίον. διαφερόντως δ' ἄν τις αὐτὸν ἀγασθείη κατὰ τὴν Γεωμετρικὴν στοιχείωσιν τῆς τάξεως ἕνεκα καὶ τῆς ἐκλογῆς τῶν πρὸς τὰ στοιχεῖα πεποιημένων θεωρημάτων τε καὶ προβλημάτων. καὶ γὰρ οὐχ ὅσα ἐνεχώρει λέγειν ἀλλ' ὅσα στοιχειοῦν ἠδύνατο παρείληφεν, ἔτι δὲ τοὺς τῶν συλλογισμῶν παντοίους τρόπους, τοὺς μὲν

---

[a] Eratosthenes was born about 284 B.C. His ability in many branches of knowledge, but failure to achieve the highest place in any, won for him the nicknames "Beta" and "Pentathlos." He became tutor to Philopator, son of Ptolemy Euergetes (see *infra*, pp. 256-257) and librarian at Alexandria. He wrote a book *Platonicus* and another *On Means* (both lost). For his *sieve* for finding successive prime numbers, see *supra*, pp. 100-103 and for his solution of the problem of doubling the cube, *infra*, pp. 290-297. His greatest achievement was his measurement of the circumference of the earth to a surprising degree of exactitude (see Heath, *H.G.M.* i. 106-108, *Greek Astronomy*, pp. 109-112).

[b] It is true that the final book of the *Elements*, as written by Euclid, dealt with the construction of the cosmic, or Platonic, figures, but the whole work was certainly not designed with a view to their construction. Euclid, however, may quite well have been a Platonist.

[c] Euclid's *Optics* survives and is available in the Teubner text in two recensions, one probably Euclid's own, the other

older than Eratosthenes and Archimedes. For these men were contemporaries, as Eratosthenes[a] somewhere says. In his aim he was a Platonist, being in sympathy with this philosophy, whence it comes that he made the end of the whole *Elements* the construction of the so-called Platonic figures.[b] There are many other mathematical writings by this man, wonderful in their accuracy and replete with scientific investigations. Such are the *Optics* and *Catoptrics*, and the *Elements of Music*, and again the book *On Divisions*.[c] He deserves admiration pre-eminently in the compilation of his *Elements of Geometry* on account of the order and of the selection both of the theorems and of the problems made with a view to the elements. For he included not everything which he could have said, but only such things as he could set down as elements. And he used all the various forms of syllogisms, some getting their plausibility from the

by Theon of Alexandria. It is possible that Proclus has attributed to Euclid a treatise on *Catoptrics* (*Mirrors*) which was really Theon's; a treatise by Euclid on this subject is not otherwise known. Two musical treatises attributed to Euclid are extant, the *Sectio Canonis* (Κατατομὴ κανόνος) and the *Introductio Harmonica* (Εἰσαγωγὴ ἁρμονική); the latter, however, is definitely by Cleonides, a pupil of Aristoxenus, and it is not certain that the former is Euclid's own. The book *On Divisions* (*of Figures*) has survived in an Arabic text discovered by Woepcke at Paris and published in 1851; see R. C. Archibald, *Euclid's Book on Division of Figures with a restoration based on Woepcke's text and the Practica Geometriae of Leonardo Pisano* (Cambridge 1915). A Latin translation (probably by Gherard of Cremona, 1114–1187) from the Arabic was known in the Middle Ages, but the Arabic cannot have been a direct translation from Euclid's Greek. The general character of the treatise is indicated by Procl. *in Eucl.* i., ed. Friedlein 144. 22-26, as the division of figures into like and unlike figures.

ἀπὸ τῶν αἰτίων λαμβάνοντας τὴν πίστιν, τοὺς δὲ ἀπὸ τεκμηρίων ὡρμημένους, πάντας δὲ ἀνελέγκτους καὶ ἀκριβεῖς καὶ πρὸς ἐπιστήμην οἰκείους, πρὸς δὲ τούτοις τὰς μεθόδους ἁπάσας τὰς διαλεκτικάς, τὴν μὲν διαιρετικὴν ἐν ταῖς εὑρέσεσι τῶν εἰδῶν, τὴν δὲ ὁριστικὴν ἐν τοῖς οὐσιώδεσι λόγοις, τὴν δὲ ἀποδεικτικὴν ἐν τοῖς ἀπὸ ἀρχῶν εἰς τὰ ζητούμενα μεταβάσεσι, τὴν δὲ ἀναλυτικὴν ἐν ταῖς ἀπὸ τῶν ζητουμένων ἐπὶ τὰς ἀρχὰς ἀναστροφαῖς. καὶ μὴν καὶ τὰ ποικίλα τῶν ἀντιστροφῶν εἴδη τῶν τε ἁπλουστέρων καὶ τῶν συνθετωτέρων ἱκανῶς ἔστιν ἐν τῇ πραγματείᾳ ταύτῃ διηκριβωμένα θεωρεῖν, καὶ τίνα μὲν ὅλα ὅλοις ἀντιστρέφειν δύναται, τίνα δὲ ὅλα μέρεσι καὶ ἀνάπαλιν, τίνα δὲ ὡς μέρη μέρεσιν. ἔτι δὲ λέγομεν τὴν συνέχειαν τῶν εὑρέσεων, τὴν οἰκονομίαν καὶ τὴν τάξιν τῶν τε προηγουμένων καὶ τῶν ἑπομένων, τὴν δύναμιν, μεθ' ἧς ἕκαστα παραδίδωσιν. ἢ καὶ τὸ τυχὸν προσθεὶς ἢ ἀφελὼν οὐκ ἐπιστήμης λανθάνεις ἀποπεσὼν καὶ εἰς τὸ ἐναντίον ψεῦδος καὶ τὴν ἄγνοιαν ὑπενεχθείς; ἐπειδὴ δὲ πολλὰ φαντάζεται μὲν ὡς τῆς ἀληθείας ἀντεχόμενα καὶ ταῖς ἐπιστημονικαῖς ἀρχαῖς ἀκολουθοῦντα, φέρεται δὲ εἰς τὴν ἀπὸ τῶν ἀρχῶν πλάνην καὶ τοὺς

---

[a] Lit. "causes," but αἴτιον clearly means the same here as ἀρχή, as often in Aristotle, *cf. Met.* Δ 1, 1013 a 16, ἰσαχῶς δὲ καὶ τὰ αἴτια λέγεται· πάντα γὰρ τὰ αἴτια ἀρχαί.

[b] Geometrical conversion is to be distinguished from logical conversion, as described by Aristotle, *Cat.* xii. 6 and elsewhere. An analysis of the conversion of geometrical propositions is given by Proclus (*in Eucl.* i., ed. Friedlein, 252. 5 *et seq.*). In the leading form of conversion (ἡ προηγουμένη ἀντιστροφή, also called conversion *par excellence*, ἡ κυρίως ἀντιστροφή) the conversion is simple, the hypo-

first principles,[a] some setting out from demonstrative proofs, all being irrefutable and accurate and in harmony with science. In addition to these he used all the dialectical methods, the *divisional* in the discovery of figures, the *definitive* in the existential arguments, the *demonstrative* in the passages from first principles to the things sought, and the *analytic* in the converse process from the things sought to the first principles. And the various species of conversions,[b] both of the simpler (propositions) and of the more complex, are in this treatise accurately set forth and skilfully investigated, what wholes can be converted with wholes, what wholes with parts and conversely, and what as parts with parts. Again, mention must be made of the continuity of the proofs, the disposition and arrangement of the things which precede and those which follow, and the power with which he treats each detail. Have you, adding or subtracting accidentally, fallen away unawares from science, carried into the opposite error and into ignorance? Since many things seem to conform with the truth and to follow from scientific principles, but lead away from the principles into error and

thesis and conclusion of one theorem becoming the conclusion and hypothesis of the converse theorem. The other form of conversion is more complex, being that where several hypotheses are combined into a single enunciation so as to lead to a single conclusion. In the converse proposition the conclusion of the original proposition is combined with the hypotheses of the original proposition, less one, so as to lead to the omitted hypothesis as the new conclusion. An example of the first species of conversion is Euclid i. 6, which is the converse of Euclid i. 5, and Heath's notes thereon are most valuable (*The Thirteen Books of Euclid's Elements*, vol. i. pp. 256-257); an example of partial conversion is given by Euclid i. 8, which is a converse to i. 4.

ἐπιπολαιοτέρους ἐξαπατᾷ, μεθόδους παραδέδωκεν καὶ τῆς τούτων διορατικῆς φρονήσεως, ἃς ἔχοντες γυμνάζειν μὲν δυνησόμεθα τοὺς ἀρχομένους τῆς θεωρίας ταύτης πρὸς τὴν εὕρεσιν τῶν παραλογισμῶν, ἀνεξαπάτητοι δὲ διαμένειν. καὶ τοῦτο δὴ τὸ σύγγραμμα, δι᾿ οὗ τὴν παρασκευὴν ἡμῖν ταύτην ἐντίθησι, Ψευδαρίων ἐπέγραψεν, τρόπους τε αὐτῶν ποικίλους ἐν τάξει διαριθμησάμενος καὶ καθ᾿ ἕκαστον γυμνάσας ἡμῶν τὴν διάνοιαν παντοίοις θεωρήμασι καὶ τῷ ψεύδει τὸ ἀληθὲς παραθεὶς καὶ τῇ πείρᾳ τὸν ἔλεγχον τῆς ἀπάτης συναρμόσας. τοῦτο μὲν οὖν τὸ βιβλίον καθαρτικόν ἐστι καὶ γυμναστικόν, ἡ δὲ Στοιχείωσις αὐτῆς τῆς ἐπιστημονικῆς θεωρίας τῶν ἐν γεωμετρίᾳ πραγμάτων ἀνέλεγκτον ἔχει καὶ τελείαν ὑφήγησιν.

deceive the more superficial, he has handed down methods for the clear-sighted understanding of these matters also, and with these methods in our possession we can train beginners in the discovery of paralogisms and avoid being misled. The treatise in which he gave this machinery to us he entitled [the book] of *Pseudaria*,[a] enumerating in order their various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of the error with practical illustration. This book is therefore purgative and disciplinary, while the *Elements* contains an irrefutable and complete guide to the actual scientific investigation of geometrical matters.

[a] This book is lost. It clearly belonged to elementary geometry.

V. THALES

V. THALES

The circle is bisected by its diameter

Procl. *in Eucl.* i., ed. Friedlein 157. 10-13

Τὸ μὲν οὖν διχοτομεῖσθαι τὸν κύκλον ὑπὸ τῆς διαμέτρου πρῶτον Θαλῆν ἐκεῖνον ἀποδεῖξαί φασιν, αἰτία δὲ τῆς διχοτομίας ἡ τῆς εὐθείας ἀπαρέγκλιτος διὰ τοῦ κέντρου χώρησις.

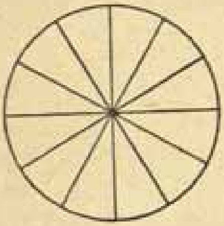
The angles at the base of an isosceles triangle are equal

Ibid. 250. 22–251. 2

Λέγεται γὰρ δὴ πρῶτος ἐκεῖνος ἐπιστῆσαι καὶ εἰπεῖν, ὡς ἄρα παντὸς ἰσοσκελοῦς αἱ πρὸς τῇ βάσει γωνίαι ἴσαι εἰσίν, ἀρχαϊκώτερον δὲ τὰς ἴσας ὁμοίας προσειρηκέναι.

---

[a] The word "demonstrate" (ἀποδεῖξαι) must not be taken too literally. Even Euclid did not demonstrate this property of the circle, but stated it as the 17th definition of his first book. Thales probably was the first to point out this property. Cantor (*Gesch. d. Math.* i³., pp. 109, 140) and Heath (*H.G.M.* i. 131) suggest that his attention may have been drawn to it by figures of circles divided into equal sectors by a number of diameters. Such figures are found on Egyptian monuments



V. THALES

The circle is bisected by its diameter

Proclus, *on Euclid* i., ed. Friedlein 157. 10-13

They say that Thales was the first to demonstrate[a] that the circle is bisected by the diameter, the cause of the bisection being the unimpeded passage of the straight line through the centre.

The angles at the base of an isosceles triangle are equal

Ibid. 250. 22–251. 2

[Thales] is said to have been the first to have known and to have enunciated [the theorem] that the angles at the base of any isosceles triangle are equal, though in the more archaic manner he described the equal angles as similar.[b]

and vessels brought by Asiatic tributary kings in the time of the eighteenth dynasty.

[b] This theorem is Eucl. i. 5, the famous *pons asinorum.* Heath notes (*H.G.M.* i. 131): "It has been suggested that the use of the word 'similar' to describe the equal angles of an isosceles triangle indicates that Thales did not yet conceive of an angle as a magnitude, but as a *figure* having a certain *shape*, a view which would agree closely with the idea of the Egyptian *se-qet*, 'that which makes the nature,' in the sense of determining a similar or the same inclination in the faces of pyramids."

The vertical and opposite angles are equal

Ibid. 299. 1-5

Τοῦτο τοίνυν τὸ θεώρημα δείκνυσιν, ὅτι δύο εὐθειῶν ἀλλήλας τεμνουσῶν αἱ κατὰ κορυφὴν γωνίαι ἴσαι εἰσίν, εὑρημένον μέν, ὥς φησιν Εὔδημος, ὑπὸ Θαλοῦ πρώτου, τῆς δὲ ἐπιστημονικῆς ἀποδείξεως ἠξιωμένου παρὰ τῷ Στοιχειωτῇ.

Equality of Triangles

Ibid. 352. 14-18

Εὔδημος δὲ ἐν ταῖς γεωμετρικαῖς ἱστορίαις εἰς Θαλῆν τοῦτο ἀνάγει τὸ θεώρημα. τὴν γὰρ τῶν ἐν θαλάττῃ πλοίων ἀπόστασιν δι' οὗ τρόπου φασὶν αὐτὸν δεικνύναι τούτῳ προσχρῆσθαί φησιν ἀναγκαῖον.

The angle in a semicircle is a right-angle

Diog. Laert. i. 24-25

Παρά τε Αἰγυπτίων γεωμετρεῖν μαθόντα φησὶ Παμφίλη πρῶτον καταγράψαι κύκλου τὸ τρίγωνον

---

[a] It is Eucl. i. 15.

[b] The method by which Thales used the theorem referred to, Eucl. i. 26, to find the distance of a ship from the shore, has given rise to many conjectures. The most attractive is that of Heath (*The Thirteen Elements of Euclid's Elements*, i., p. 305, *H.G.M.* i. 133). He supposes that the observer had a rough instrument made of a straight stick and a cross-piece fastened to it so as to be capable of turning about the

The vertical and opposite angles are equal

Ibid. 299. 1-5

This theorem, that when two straight lines cut one another the vertical and opposite angles are equal, was first discovered, as Eudemus says, by Thales, though the scientific demonstration was improved by the writer of the *Elements*.[a]

Equality of Triangles

Ibid. 352. 14-18

Eudemus in his *History of Geometry* attributes this theorem to Thales. For he says that the method by which Thales showed how to find the distance of ships at sea necessarily involves this method.[b]

The angle in a semicircle is a right-angle

Diogenes Laertius i. 24-25

Pamphila says that, having learnt geometry from the Egyptians, he was the first to inscribe in a circle

fastening in such a manner so that it could form any angle with the stick and would remain where it was put. The observer, standing on the top of a tower or some other eminence on the shore, would fix the stick in the upright position and direct the cross-piece towards the ship. Leaving the cross-piece at this angle, he would turn the stick round, keeping it vertical, until the cross-piece pointed to some object on the land, which would be noted. The distance between the foot of the tower and this object would, by Eucl. i. 26, be equal to the distance of the ship. Apparently this method is found in many practical geometries during the first century of printing.

ὀρθογώνιον, καὶ θῦσαι βοῦν. οἱ δὲ Πυθαγόραν φασίν, ὧν ἐστιν Ἀπολλόδωρος ὁ λογιστικός.

ᵃ Pamphila was a female writer who lived in the reign of Nero and won much repute by her historical commonplace book (*Συμμίκτων ἱστορικῶν ὑπομνημάτων λόγοι*). She may have been right in ascribing to Thales the discovery that the angle in a semicircle is a right angle, but the passage bristles with difficulties. The reference to the sacrifice of an ox is suspiciously like the better-attested story that Pythagoras sacrificed oxen when he discovered a certain theorem. This story is told in a distich by Apollodorus reproduced below (p. 176). In reproducing that distich Plutarch says it is uncertain whether the theorem was that about the square on the hypotenuse of a right-angled triangle or that about the application of areas: he does not mention the theorem about the angle in a semicircle. Diogenes Laertius probably made a mistake in bringing in Apollodorus: the reference to the sacrifice of an ox made him think of Apollodorus's distich

a right-angled triangle, whereupon he sacrificed an ox. Others say it was Pythagoras, among them being Apollodorus the calculator.[a]

about Pythagoras, forgetting that they referred to a different proposition.

There are also difficulties on the way of believing that Thales could have discovered the theorem that the angle in a semicircle is a right angle. Euclid (iii. 31) proves this theorem by means of i. 32, that the sum of the angles of any triangle is two right-angles. Now Eudemus, as will be found below, pp. 176-179, attributed to the Pythagoreans the discovery of the theorem that in any triangle the sum of the angles is equal to two right-angles. The authority of Eudemus compels us to believe that Thales did not know this theorem. Could he have proved that the angle in a semicircle is a right angle without previously knowing that the sum of the angles of any triangle is two right-angles? Heath (*H.G.M.* i. 136-137) shows how he could have done so; and so Pamphila, for all her late date, may have preserved a correct tradition.

VI. PYTHAGOREAN GEOMETRY

VI. PYTHAGOREAN GEOMETRY

(a) General

Apollon. *Mirab.* 6 : Diels, *Vors.* i[5]. 98. 29-31

Πυθαγόρας Μνησάρχου υἱὸς τὸ μὲν πρῶτον διεπονεῖτο περὶ τὰ μαθήματα καὶ τοὺς ἀριθμούς, ὕστερον δέ ποτε καὶ τῆς Φερεκύδου τερατοποιίας οὐκ ἀπέστη.

Aristot. *Met.* A 5, 985 b 23-26

Ἐν δὲ τούτοις καὶ πρὸ τούτων οἱ καλούμενοι Πυθαγόρειοι τῶν μαθημάτων ἁψάμενοι πρῶτοι ταῦτά τε προήγαγον, καὶ ἐντραφέντες ἐν αὐτοῖς τὰς τούτων ἀρχὰς τῶν ὄντων ἀρχὰς ᾠήθησαν εἶναι πάντων.

Diog. Laert. viii. 24-25

Φησὶ δ' ὁ Ἀλέξανδρος ἐν ταῖς τῶν φιλοσόφων διαδοχαῖς καὶ ταῦτα εὑρηκέναι ἐν Πυθαγορικοῖς ὑπομνήμασιν. ἀρχὴν μὲν ἁπάντων μονάδα· ἐκ δὲ τῆς μονάδος ἀόριστον δυάδα ὡς ἂν ὕλην τῇ μονάδι αἰτίῳ ὄντι ὑποστῆναι· ἐκ δὲ τῆς μονάδος καὶ τῆς ἀορίστου δυάδος τοὺς ἀριθμούς· ἐκ δὲ τῶν ἀριθμῶν τὰ σημεῖα· ἐκ δὲ τούτων τὰς γραμμάς, ἐξ ὧν τὰ

VI. PYTHAGOREAN GEOMETRY

(a) General

Apollonius Paradoxographus, *On Marvels* 6; Diels, *Vors.* i[5]. 98. 29-31 [a]

PYTHAGORAS, the son of Mnesarchus, first worked at mathematics and numbers, and later at one time did not hold himself aloof from the wonder-working of Pherecydes.

Aristotle, *Metaphysics* A 5, 985 b 23-26

In the time of these men [Leucippus and Democritus] and before them the so-called Pythagoreans applied themselves to mathematics and were the first to advance that science; and because they had been brought up in it they thought that its principles must be the principles of all existing things.

Diogenes Laertius viii. 24-25

Alexander in *The Successions of Philosophers* says that he found in the Pythagorean memoirs these beliefs also. The principle of all things is the monad; arising from the monad, the undetermined dyad acts as matter to the monad, which is cause; from the monad and the undetermined dyad arise numbers; from numbers, points; from these, lines, out of which

[a] Apollonius is quoting Aristotle's book *On the Pythagoreans*, now lost.

ἐπίπεδα σχήματα· ἐκ δὲ τῶν ἐπιπέδων τὰ στερεὰ σχήματα· ἐκ δὲ τούτων τὰ αἰσθητὰ σώματα, ὧν καὶ τὰ στοιχεῖα εἶναι τέτταρα, πῦρ, ὕδωρ, γῆν, ἀέρα· μεταβάλλειν δὲ καὶ τρέπεσθαι δι' ὅλων, καὶ γίνεσθαι ἐξ αὐτῶν κόσμον ἔμψυχο', νοερόν, σφαιροειδῆ, μέσην περιέχοντα τὴν γῆν καὶ αὐτὴν σφαιροειδῆ καὶ περιοικουμένην.

Diog. Laert. viii. 11-12

Τοῦτον καὶ γεωμετρίαν ἐπὶ πέρας ἀγαγεῖν, Μοίριδος πρώτου εὑρόντος τὰς ἀρχὰς τῶν στοιχείων αὐτῆς, ὥς φησιν Ἀντικλείδης ἐν δευτέρῳ Περὶ Ἀλεξάνδρου. μάλιστα δὲ σχολάσαι τὸν Πυθαγόραν περὶ τὸ ἀριθμητικὸν εἶδος αὐτῆς τόν τε κανόνα τὸν ἐκ μιᾶς χορδῆς εὑρεῖν. οὐκ ἠμέλησε δ' οὐδ' ἰατρικῆς. φησὶ δ' Ἀπολλόδωρος ὁ λογιστικὸς ἑκατόμβην θῦσαι αὐτόν, εὑρόντα ὅτι τοῦ ὀρθογωνίου τριγώνου ἡ ὑποτείνουσα πλευρὰ ἴσον δύναται ταῖς περιεχούσαις. καὶ ἔστιν ἐπίγραμμα οὕτως ἔχον·

ἡνίκα Πυθαγόρης τὸ περικλεὲς εὕρετο γράμμα,
κεῖν' ἐφ' ὅτῳ κλεινὴν ἤγαγε βουθυσίην.

Procl. in *Eucl.* i., ed. Friedlein 84. 13-23

Ὅσα δὲ πραγματειωδεστέραν ἔχει θεωρίαν καὶ συντελεῖ πρὸς τὴν ὅλην φιλοσοφίαν, τούτων προηγουμένην ποιησόμεθα τὴν ὑπόμνησιν, ζηλοῦντες τοὺς Πυθαγορείους, οἷς πρόχειρον ἦν καὶ τοῦτο σύμβολον "σχᾶμα καὶ βᾶμα, ἀλλ' οὐ σχᾶμα καὶ τριώβολον" ἐνδεικνυμένων, ὡς ἄρα δεῖ τὴν γεωμετρίαν ἐκείνην μεταδιώκειν, ἣ καθ' ἕκαστον

arise plane figures; from planes, solid figures; from these, sensible bodies, whose elements are four—fire, water, earth, air; these elements interchange and turn into one another completely, and out of them arises a world which is animate, intelligent, spherical, and having as its centre the earth, which also is spherical and is inhabited round about.

Diogenes Laertius viii. 11-12

He [Pythagoras] it was who brought geometry to perfection, after Moeris had first discovered the beginnings of the elements of that science, as Anticleides says in the second book of his *History of Alexander*. He adds that Pythagoras specially applied himself to the arithmetical aspect of geometry and he discovered the musical intervals on the monochord; nor did he neglect even medicine. Apollodorus the calculator says that he sacrificed a hecatomb on finding that the square on the hypotenuse of the right-angled triangle is equal to the squares on the sides containing the right angle. And there is an epigram as follows:

> As when Pythagoras the famous figure found,
> For which a sacrifice renowned he brought.

Proclus, *on Euclid* i., ed. Friedlein 84. 13-23

Whatsoever offers a more profitable field of research and contributes to the whole of philosophy, we shall make the starting-point of further inquiry, therein imitating the Pythagoreans, among whom there was prevalent this motto, "A figure and a platform, not a figure and sixpence," by which they implied that the geometry deserving study is that which, at each

θεώρημα βῆμα τίθησιν εἰς ἄνοδον καὶ ἀπαίρει τὴν ψυχὴν εἰς ὕψος, ἀλλ' οὐκ ἐν τοῖς αἰσθητοῖς καταβαίνειν ἀφίησιν καὶ τὴν σύνοικον τοῖς θνητοῖς χρείαν ἀποπληροῦν καὶ ταύτης στοχαζομένην τῆς ἐντεῦθεν περιαγωγῆς καταμελεῖν.

Plut. *Non posse suav. vivi sec. Epic.* 11, 1094 B

Καὶ Πυθαγόρας ἐπὶ τῷ διαγράμματι βοῦν ἔθυσεν, ὥς φησιν Ἀπολλόδωρος·

ἡνίκα Πυθαγόρας τὸ περικλεὲς εὕρετο γράμμα
κεῖν' ἐφ' ὅτῳ λαμπρὴν ἤγετο βουθυσίην.

εἴτε περὶ τῆς ὑποτεινούσης ὡς ἴσον δύναται ταῖς περιεχούσαις τὴν ὀρθήν, εἴτε πρόβλημα περὶ τοῦ χωρίου τῆς παραβολῆς.

Plut. *Quaest. Conv.* viii. 2. 4, 720 A

Ἔστι γὰρ ἐν τοῖς γεωμετρικωτάτοις θεωρήμασι, μᾶλλον δὲ προβλήμασι, τὸ δυεῖν εἰδῶν δοθέντων ἄλλο τρίτον παραβάλλειν τῷ μὲν ἴσον τῷ δ' ὅμοιον· ἐφ' ᾧ καί φασιν ἐξευρεθέντι θῦσαι τὸν Πυθαγόραν. πολὺ γὰρ ἀμέλει γλαφυρώτερον τοῦτο καὶ μουσικώτερον ἐκείνου τοῦ θεωρήματος, ὃ τὴν ὑποτείνουσαν ἀπέδειξε ταῖς περὶ τὴν ὀρθὴν ἴσον δυναμένην.

(b) Sum of the Angles of a Triangle

Procl. *in Eucl.* i., ed. Friedlein 379. 2-16

Εὔδημος δὲ ὁ περιπατητικὸς εἰς τοὺς Πυθαγορείους ἀναπέμπει τὴν τοῦδε τοῦ θεωρήματος εὕρεσιν, ὅτι τρίγωνον ἅπαν δυσὶν ὀρθαῖς ἴσας ἔχει

theorem, sets up a platform for further ascent and lifts the soul on high, instead of allowing it to descend among sensible objects and so fulfil the common needs of mortal men and in this lower aim neglect conversion to things above.

Plutarch, *The Epicurean Life* 11, 1094 B

Pythagoras sacrificed an ox in virtue of his proposition, as Apollodorus says—

As when Pythagoras the famous figure found
For which the noble sacrifice he brought [a]—

whether it was the theorem that the square on the hypotenuse is equa- to the squares on the sides containing the right angle, or the problem about the application of the area.

Plutarch, *Convivial Questions* viii. 2. 4, 720 A

Among the most geometrical theorems, or rather problems, is this—given two figures, to apply a third equal to the one and similar to the other; it was in virtue of this discovery they say Pythagoras sacrificed. This is unquestionably more subtle and elegant than the theorem which he proved that the square on the hypotenuse is equal to the squares on the sides about the right angle.

(*b*) Sum of the Angles of a Triangle

Proclus, *on Euclid* i., ed. Friedlein 379. 2-16

Eudemus the Peripatetic ascribes to the Pythagoreans the discovery of this theorem, that any triangle has its internal angles equal to two right

[a] See *supra*, p. 168 n. *a*, and p. 174.

τὰς ἐντὸς γωνίας. καὶ δεικνύναι φησὶν αὐτοὺς οὕτως τὸ προκείμενον. ἔστω τρίγωνον τὸ ΑΒΓ, καὶ ἤχθω διὰ τοῦ Α τῇ ΒΓ παράλληλος ἡ ΔΕ. ἐπεὶ οὖν παράλληλοί εἰσιν αἱ ΒΓ, ΔΕ, καὶ αἱ ἐναλλὰξ ἴσαι εἰσίν, ἴση ἄρα ἡ μὲν ὑπὸ ΔΑΒ τῇ ὑπὸ ΑΒΓ, ἡ δὲ ὑπὸ ΕΑΓ τῇ ὑπὸ ΑΓΒ. κοινὴ προσκείσθω ἡ ΒΑΓ. αἱ ἄρα ὑπὸ ΔΑΒ, ΒΑΓ, ΓΑΕ, τουτέστιν αἱ ὑπὸ ΔΑΒ, ΒΑΕ, τουτέστιν αἱ δύο ὀρθαὶ ἴσαι εἰσὶ ταῖς τοῦ ΑΒΓ τριγώνου τρισὶ γωνίαις. αἱ ἄρα τρεῖς τοῦ τριγώνου δύσιν ὀρθαῖς εἰσιν ἴσαι.

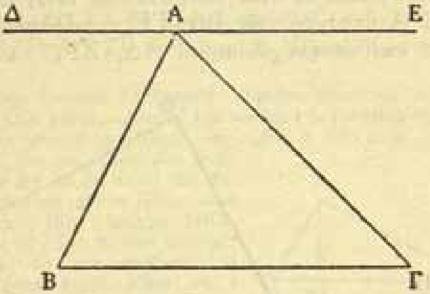
(c) "Pythagoras's Theorem"

Eucl. *Elem.* i. 47

Ἐν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις.

Ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ ὀρθὴν ἔχον τὴν ὑπὸ ΒΑΓ γωνίαν· λέγω ὅτι τὸ ἀπὸ τῆς ΒΓ τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ τετραγώνοις.

angles. He says they proved the theorem in question



after this fashion. Let ΑΒΓ be a triangle, and through A let ΔΕ be drawn parallel to ΒΓ. Now since ΒΓ, ΔΕ are parallel, and the alternate angles are equal, the angle ΔΑΒ is equal to the angle ΑΒΓ, and ΕΑΓ is equal to ΑΓΒ. Let ΒΑΓ be added to both. Then the angles ΔΑΒ, ΒΑΓ, ΓΑΕ, that is, the angles ΔΑΒ, ΒΑΕ, that is, two right angles, are equal to the three angles of the triangle. Therefore the three angles of the triangle are equal to two right angles.

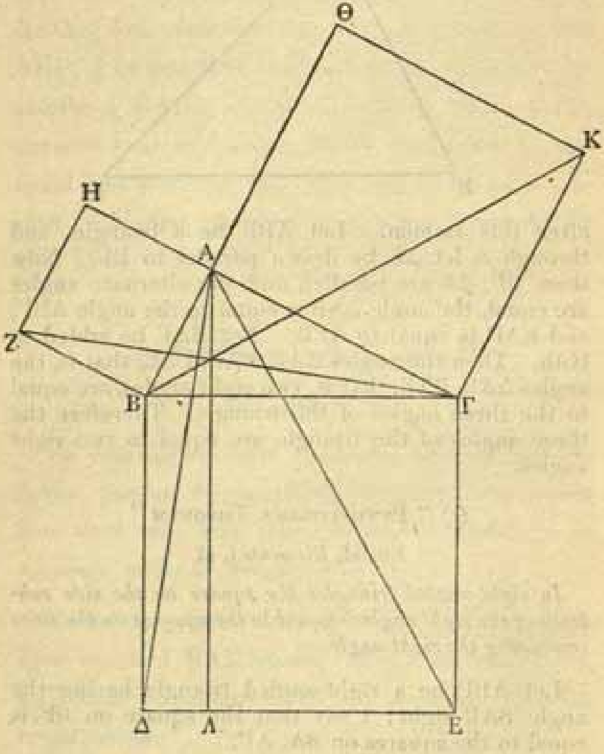
(c) " Pythagoras's Theorem "

Euclid, *Elements* i. 47

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Let ΑΒΓ be a right-angled triangle having the angle ΒΑΓ right; I say that the square on ΒΓ is equal to the squares on ΒΑ, ΑΓ.

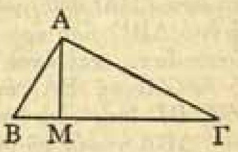
Ἀναγεγράφθω γὰρ ἀπὸ μὲν τῆς ΒΓ τετράγωνον τὸ ΒΔΕΓ, ἀπὸ δὲ τῶν ΒΑ, ΑΓ τὰ ΗΒ, ΘΓ, καὶ διὰ τοῦ Α ὁποτέρᾳ τῶν ΒΔ, ΓΕ παράλληλος ἤχθω ἡ ΑΛ· καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΖΓ. καὶ ἐπεὶ



For let there be described on ΒΓ the square ΒΔΕΓ, and on ΒΑ, ΑΓ the squares ΗΒ, ΘΓ [Eucl. i. 46], and through Α let ΑΛ be drawn parallel to either ΒΔ or ΓΕ, and let ΑΔ, ΖΓ be joined.[a] Then, since each of

[a] In this famous "windmill" figure, the lines ΑΛ, ΒΚ, ΓΖ meet in a point. Euclid has no need to mention this fact, but it was proved by Heron; see *infra*, p. 185 n. *b*.

If ΑΛ, the perpendicular from Α, meets ΒΓ in Μ, as in the detached portion of the figure here reproduced, the triangles ΜΒΑ, ΜΑΓ are similar to the triangle ΑΒΓ and to one another. It follows from Eucl. *Elem.* vi. 4 and 17 (which do not depend on i. 47) that



$$\mathrm{BA}^2 = \mathrm{BM} \,.\, \mathrm{B\Gamma},$$

and $$\mathrm{A\Gamma}^2 = \mathrm{\Gamma M} \,.\, \mathrm{B\Gamma}.$$

Therefore $$\mathrm{BA}^2 + \mathrm{A\Gamma}^2 = \mathrm{B\Gamma}\,(\mathrm{BM} + \mathrm{\Gamma M}) = \mathrm{B\Gamma}^2.$$

The theory of proportion developed in Euclid's sixth book therefore offers a simple method of proving "Pythagoras's Theorem." This proof, moreover, is of the same type as Eucl. *Elem.* i. 47 inasmuch as it is based on the equality of the square on ΒΓ to the sum of two rectangles. This has suggested that Pythagoras proved the theorem by means of his inadequate theory of proportion, which applied only to commensurable magnitudes. When the incommensurable was discovered, it became necessary to find a new proof independent of proportions. Euclid therefore recast Pythagoras's invalidated proof in the form here given so as to get it into the first book in accordance with his general plan of the *Elements*.

For other methods by which the theorem can be proved, the complete evidence bearing on its reputed discovery by Pythagoras, and the history of the theorem in Egypt, Babylonia, and India, see Heath, *The Thirteen Books of Euclid's Elements*, i., pp. 351-366, *A Manual of Greek Mathematics*, pp. 95-100.

ὀρθή ἐστιν ἑκατέρα τῶν ὑπὸ ΒΑΓ, ΒΑΗ γωνιῶν, πρὸς δή τινι εὐθείᾳ τῇ ΒΑ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α δύο εὐθεῖαι αἱ ΑΓ, ΑΗ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιοῦσιν· ἐπ' εὐθείας ἄρα ἐστὶν ἡ ΓΑ τῇ ΑΗ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΒΑ τῇ ΑΘ ἐστιν ἐπ' εὐθείας. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΒΓ γωνία τῇ ὑπὸ ΖΒΑ· ὀρθὴ γὰρ ἑκατέρα· κοινὴ προσκείσθω ἡ ὑπὸ ΑΒΓ· ὅλη ἄρα ἡ ὑπὸ ΔΒΑ ὅλῃ τῇ ὑπὸ ΖΒΓ ἐστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΒ τῇ ΒΓ, ἡ δὲ ΖΒ τῇ ΒΑ, δύο δὴ αἱ ΔΒ, ΒΑ δύο ταῖς ΖΒ, ΒΓ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΔΒΑ γωνίᾳ τῇ ὑπὸ ΖΒΓ ἴση· βάσις ἄρα ἡ ΑΔ βάσει τῇ ΖΓ [ἐστιν] ἴση, καὶ τὸ ΑΒΔ τρίγωνον τῷ ΖΒΓ τριγώνῳ ἐστὶν ἴσον· καί [ἐστι] τοῦ μὲν ΑΒΔ τριγώνου διπλάσιον τὸ ΒΛ παραλληλόγραμμον· βάσιν τε γὰρ τὴν αὐτὴν ἔχουσι τὴν ΒΔ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ΒΔ, ΑΛ· τοῦ δὲ ΖΒΓ τριγώνου διπλάσιον τὸ ΗΒ τετράγωνον· βάσιν τε γὰρ πάλιν τὴν αὐτὴν ἔχουσι τὴν ΖΒ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ΖΒ, ΗΓ. [τὰ δὲ τῶν ἴσων διπλάσια ἴσα ἀλλήλοις ἐστίν·][1] ἴσον ἄρα ἐστὶ καὶ τὸ ΒΛ παραλληλόγραμμον τῷ ΗΒ τετραγώνῳ. ὁμοίως δὴ ἐπιζευγνυμένων τῶν ΑΕ, ΒΚ δειχθήσεται καὶ τὸ ΓΛ παραλληλόγραμμον ἴσον τῷ ΘΓ τετραγώνῳ· ὅλον ἄρα τὸ ΒΔΕΓ τετράγωνον δυσὶ τοῖς ΗΒ, ΘΓ τετραγώνοις ἴσον ἐστίν. καί ἐστι τὸ μὲν ΒΔΕΓ τετράγωνον ἀπὸ τῆς ΒΓ ἀναγραφέν, τὰ δὲ ΗΒ, ΘΓ ἀπὸ τῶν ΒΑ, ΑΓ. τὸ ἄρα ἀπὸ τῆς ΒΓ πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ πλευρῶν τετραγώνοις.

Ἐν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς

the angles ΒΑΓ, ΒΑΗ is right, it follows that with a straight line ΒΑ and at the point Α on it, two straight lines ΑΓ, ΑΗ, not lying on the same side, make the adjacent angles equal to two right angles; therefore ΓΑ is in a straight line with ΑΗ [Eucl. i. 14]. For the same reasons ΒΑ is also in a straight line with ΑΘ. And since the angle ΔΒΓ is equal to the angle ΖΒΑ, for each is right, let the angle ΑΒΓ be added to each; the whole angle ΔΒΑ is therefore equal to the whole angle ΖΒΓ. And since ΔΒ is equal to ΒΓ, and ΖΒ to ΒΑ, the two [sides] ΔΒ, ΒΑ are equal to the two [sides] ΒΓ, ΖΒ respectively; and the angle ΔΒΑ is equal to the angle ΖΒΓ. The base ΑΔ is therefore equal to the base ΖΓ, and the triangle ΑΒΔ is equal to the triangle ΖΒΓ [Eucl. i. 4]. Now the parallelogram ΒΛ is double the triangle ΑΒΔ, for they have the same base ΒΔ and are in the same parallels ΒΔ, ΑΛ [Eucl. i. 41]. And the square ΗΒ is double the triangle ΖΒΓ, for they have the same base ΖΒ and are in the same parallels ΖΒ, ΗΓ. Therefore the parallelogram ΒΛ is equal to the square ΗΒ. Similarly, if ΑΕ, ΒΚ are joined, it can also be proved that the parallelogram ΓΛ is equal to the square ΘΓ. Therefore the whole square ΒΔΕΓ is equal to the two squares ΗΒ, ΘΓ. And the square ΒΔΕΓ is described on ΒΓ, while the squares ΗΒ, ΘΓ are described on ΒΑ, ΑΓ. Therefore the square on the side ΒΓ is equal to the squares on the sides ΒΑ, ΑΓ.

Therefore in right-angled triangles the square on

---

[1] om. Heiberg. The words are equivalent to Common Notion 5, which must also be an interpolation as it is covered by Common Notion 2, *καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα*, "if equals are added to equals the wholes are equal."

τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν [γωνίαν] περιεχουσῶν πλευρῶν τετραγώνοις· ὅπερ ἔδει δεῖξαι.

Procl. *in Eucl.* i., ed. Friedlein 426. 6-14

Τῶν μὲν ἱστορεῖν τὰ ἀρχαῖα βουλομένων ἀκουόντας τὸ θεώρημα τοῦτο εἰς Πυθαγόραν ἀναπεμπόντων ἐστὶν εὑρεῖν καὶ βουθύτην λεγόντων αὐτὸν ἐπὶ τῇ εὑρέσει. ἐγὼ δὲ θαυμάζω μὲν καὶ τοὺς πρώτους ἐπιστάντας τῇ τοῦδε τοῦ θεωρήματος ἀληθείᾳ, μειζόνως δὲ ἄγαμαι τὸν Στοιχειωτήν, οὐ μόνον ὅτι δι' ἀποδείξεως ἐναργεστάτης τοῦτο κατεδήσατο, ἀλλ' ὅτι καὶ τὸ καθολικώτερον αὐτοῦ τοῖς ἀνελέγκτοις λόγοις τῆς ἐπιστήμης ἐπίεσεν ἐν τῷ ἕκτῳ βιβλίῳ.

Ibid. 429. 9-15

Τῆς δὲ τοῦ Στοιχειωτοῦ ἀποδείξεως οὔσης φανερᾶς οὐδὲν ἡγοῦμαι δεῖν προσθεῖναι περιττόν, ἀλλὰ ἀρκεῖσθαι τοῖς γεγραμμένοις, ἐπεὶ καὶ ὅσοι προσέθεσάν τι πλεόν, ὡς οἱ περὶ Ἥρωνα καὶ Πάππον, ἠναγκάσθησαν προσλαβεῖν τι τῶν ἐν τῷ ἕκτῳ δεδειγμένων, οὐδενὸς ἕνεκα πραγματειώδους.

---

[a] Eucl. vi. 31. *In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.*

[b] By *οἱ περὶ Ἥρωνα καὶ Πάππον* Proclus doubtless means, in accordance with his practice elsewhere, Heron and Pappus themselves. Pappus, in *Coll.* iv. 1, ed. Hultsch 176-178,

the side subtending the right angle is equal to the squares on the sides containing the right angle; which was to be proved.

Proclus, *on Euclid* i., ed. Friedlein 426. 6-14

If we listen to those who wish to relate ancient history, we find some of them attributing this theorem to Pythagoras and saying that he sacrificed an ox upon the discovery. For my part, while I admire those who first became acquainted with the truth of this theorem, I marvel more at the writer of the *Elements*, not only because he established it by a most lucid demonstration, but because he insisted on the more general theorem by the irrefutable arguments of science in the sixth book.[a]

Ibid. 429. 9-15

The proof by the writer of the *Elements* being clear, I think that it is unnecessary to add anything further, and that we may be content with what has been written, since, in fact, those who have added anything more, such as Heron and Pappus, were compelled to make use of what is proved in the sixth book, with no real object.[b]

generalized "Pythagoras's Theorem" by proving that if *any* triangle is taken (not necessarily right-angled), and *any parallelograms* are described on two of the sides, their sum is equal to a third parallelogram. Proclus's words can, however, hardly refer to this elegant theorem. Heron is known from the Arabic commentary of an-Nairizi on Euclid's *Elements* (ed. Besthorn-Heiberg 175-185) to have proved that in Euclid's figure AΛ, BK, ΓZ meet in a point. Heron used three lemmas proved on the principles of Book i. alone, but they would more easily be proved from Book vi. It is quite likely that Proclus refers to this proof.

(*d*) The Application of Areas

One of the greatest of Pythagorean discoveries was the method known as the application of areas, which became a powerful engine in the hands of successive Greek geometers. The geometer is said *to apply* (παραβάλλειν) an area to a given straight line when a rectangle or parallelogram equal to the area is constructed on that straight line exactly ; the area is said *to fall short* or *be deficient* (ἐλλείπειν) when the rectangle or parallelogram is constructed on a portion of the straight line ; and *to exceed* (ὑπερβάλλειν) when the rectangle or parallelogram is constructed on the straight line produced. The method is developed in the following propositions of Euclid's *Elements* : i. 44, 45 ; ii. 5, 6, 11 ; vi. 27, 28, 29. These proposi-

Procl. *in Eucl.* i., ed. Friedlein 419. 15–420. 12

Ἔστι μὲν ἀρχαῖα, φασὶν οἱ περὶ τὸν Εὔδημον, καὶ τῆς τῶν Πυθαγορείων μούσης εὑρήματα ταῦτα, ἥ τε παραβολὴ τῶν χωρίων καὶ ἡ ὑπερβολὴ καὶ ἡ ἔλλειψις. ἀπὸ δὲ τούτων καὶ οἱ νεώτεροι τὰ ὀνόματα λαβόντες μετήγαγον αὐτὰ καὶ ἐπὶ τὰς κωνικὰς λεγομένας γραμμάς, καὶ τούτων τὴν μὲν παραβολήν, τὴν δὲ ὑπερβολὴν καλέσαντες, τὴν δὲ

tions are equivalent to the solution of quadratic equations, not only in particular cases but in the most general form. The application of areas (παραβολὴ τῶν χωρίων) is therefore a vital part of the "geometrical algebra" of the Greeks, who dealt in figures as familiarly as we do in symbols. This method is the foundation of Euclid's theory of irrationals and Apollonius's treatment of the conic sections. The subject will be introduced by Proclus's comment on Eucl. i. 44, and then the relevant propositions of Euclid will be given, with their equivalents in modern algebraical notation. Though the precise form of the later propositions cannot be due to Pythagoras, depending as they do on a theory of proportion invented by Eudoxus, there can be no doubt, as Eudemus said, that the method goes back to the Pythagorean school, and most probably to the master himself.

Proclus, *on Euclid* i., ed. Friedlein 419. 15–420. 12

These things are ancient, says Eudemus, being discoveries of the Muse of the Pythagoreans, I mean the application of areas, their exceeding and their falling short. From these men the more recent geometers took the names that they gave to the so-called conic lines, calling one of these the *parabola*, one the *hyperbola* and one

ἔλλειψιν, ἐκείνων τῶν παλαιῶν καὶ θείων ἀνδρῶν ἐν ἐπιπέδῳ καταγραφῇ χωρίων πρὸς εὐθεῖαν ὡρισμένην τὰ ὑπὸ τούτων σημαινόμενα τῶν ὀνομάτων ὁρώντων. ὅταν γὰρ εὐθείας ἐκκειμένης τὸ δοθὲν χωρίον πάσῃ τῇ εὐθείᾳ συμπαρατείνῃς, τότε παραβάλλειν ἐκεῖνο τὸ χωρίον φασίν, ὅταν μεῖζον δὲ ποιήσῃς τοῦ χωρίου τὸ μῆκος αὐτῆς τῆς εὐθείας, τότε ὑπερβάλλειν, ὅταν δὲ ἔλασσον, ὡς τοῦ χωρίου γραφέντος εἶναί τι τῆς εὐθείας ἐκτός, τότε ἐλλείπειν. καὶ οὕτως ἐν τῷ ἕκτῳ βιβλίῳ καὶ τῆς ὑπερβολῆς ὁ Εὐκλείδης μνημονεύει καὶ τῆς ἐλλείψεως, ἐνταῦθα δὲ τῆς παραβολῆς ἐδεήθη τῷ δοθέντι τριγώνῳ παρὰ τὴν δοθεῖσαν εὐθεῖαν ἴσον ἐθέλων παραβαλεῖν [παραλληλόγραμμον], ἵνα μὴ μόνον σύστασιν ἔχωμεν παραλληλογράμμου τῷ δοθέντι τριγώνῳ ἴσον, ἀλλὰ καὶ παρ᾿ εὐθεῖαν ὡρισμένην παραβολήν.

Eucl. *Elem.* i. 44

Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι τριγώνῳ ἴσον παραλληλόγραμμον παραβαλεῖν ἐν τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.

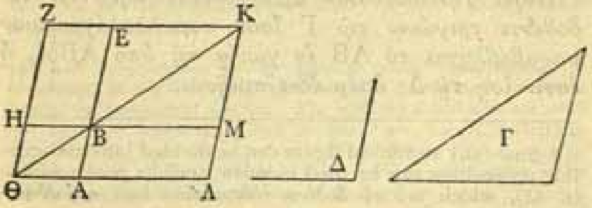
Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν τρίγωνον τὸ Γ, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ Δ· δεῖ δὴ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι τριγώνῳ τῷ Γ ἴσον παραλληλόγραμμον παραβαλεῖν ἐν ἴσῃ τῇ Δ γωνίᾳ.

the *ellipse*, inasmuch as those god-like men of old saw the things signified by these names in the construction, in a plane, of areas upon a finite straight line. For when a straight line is set out and you lay the given area exactly alongside the whole of the straight line, they say that you apply that area; but when you make the length of the area greater than the straight line, then it is said to exceed, and when you make it less, so that when the area is drawn a portion of the straight line extends beyond it, it is said to fall short. In the sixth book Euclid speaks in this way both of exceeding and of falling short, but here he needed only the application, as he sought to apply to the given straight line an area equal to the given triangle, in order that we might have not only the construction of a parallelogram equal to the given triangle, but also its application to a finite straight line.

Euclid, *Elements* i. 44

To a given straight line to apply in a given rectilineal angle a parallelogram equal to a given triangle.

Let AB be the given straight line, Γ the given triangle and Δ the given rectilineal angle; then it is required to apply to the given straight line AB, in an angle equal to the angle Δ, a parallelogram equal to the given triangle Γ.



Συνεστάτω τῷ Γ τριγώνῳ ἴσον παραλληλόγραμμον τὸ ΒΕΖΗ ἐν γωνίᾳ τῇ ὑπὸ ΕΒΗ, ἥ ἐστιν ἴση τῇ Δ· καὶ κείσθω ὥστε ἐπ' εὐθείας εἶναι τὴν ΒΕ τῇ ΑΒ, καὶ διήχθω ἡ ΖΗ ἐπὶ τὸ Θ, καὶ διὰ τοῦ Α ὁποτέρᾳ τῶν ΒΗ, ΕΖ παράλληλος ἤχθω ἡ ΑΘ, καὶ ἐπεζεύχθω ἡ ΘΒ. καὶ ἐπεὶ εἰς παραλλήλους τὰς ΑΘ, ΕΖ εὐθεῖα ἐνέπεσεν ἡ ΘΖ, αἱ ἄρα ὑπὸ ΑΘΖ, ΘΖΕ γωνίαι δυσὶν ὀρθαῖς εἰσιν ἴσαι. αἱ ἄρα ὑπὸ ΒΘΗ, ΗΖΕ δύο ὀρθῶν ἐλάσσονές εἰσιν· αἱ δὲ ἀπὸ ἐλασσόνων ἢ δύο ὀρθῶν εἰς ἄπειρον ἐκβαλλόμεναι συμπίπτουσιν· αἱ ΘΒ, ΖΕ ἄρα ἐκβαλλόμεναι συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Κ, καὶ διὰ τοῦ Κ σημείου ὁποτέρᾳ τῶν ΕΑ, ΖΘ παράλληλος ἤχθω ἡ ΚΛ, καὶ ἐκβεβλήσθωσαν αἱ ΘΑ, ΗΒ ἐπὶ τὰ Λ, Μ σημεῖα. παραλληλόγραμμον ἄρα ἐστὶ τὸ ΘΛΚΖ, διάμετρος δὲ αὐτοῦ ἡ ΘΚ, περὶ δὲ τὴν ΘΚ παραλληλόγραμμα μὲν τὰ ΑΗ, ΜΕ, τὰ δὲ λεγόμενα παραπληρώματα τὰ ΛΒ, ΒΖ· ἴσον ἄρα ἐστὶ τὸ ΛΒ τῷ ΒΖ. ἀλλὰ τὸ ΒΖ τῷ Γ τριγώνῳ ἐστὶν ἴσον· καὶ τὸ ΛΒ ἄρα τῷ Γ ἐστιν ἴσον. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΗΒΕ γωνία τῇ ὑπὸ ΑΒΜ, ἀλλὰ ἡ ὑπὸ ΗΒΕ τῇ Δ ἐστιν ἴση, καὶ ἡ ὑπὸ ΑΒΜ ἄρα τῇ Δ γωνίᾳ ἐστὶν ἴση.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν ΑΒ τῷ δοθέντι τριγώνῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΛΒ ἐν γωνίᾳ τῇ ὑπὸ ΑΒΜ, ἥ ἐστιν ἴση τῇ Δ· ὅπερ ἔδει ποιῆσαι.

---

[a] Since any rectilineal figure can be divided into triangles, this proposition can be used to solve Euclid's next problem (i. 45), which is: τῷ δοθέντι εὐθυγράμμῳ ἴσον παραλληλό-

Let the parallelogram ΒΕΖΗ be constructed, equal to the triangle Γ, in the angle ΕΒΗ which is equal to Δ [i. 42]; and let it be placed so that ΒΕ is in a straight line with ΑΒ, and let ΖΗ be produced to Θ, and through Α let ΑΘ be drawn parallel to either ΒΗ or ΕΖ [i. 31], and let ΘΒ be joined. Then, since the straight line ΘΖ falls upon the parallels ΑΘ, ΕΖ, the angles ΑΘΖ, ΘΖΕ are equal to two right angles [i. 29]. Therefore the angles ΒΘΗ, ΗΖΕ are less than two right angles. Now the straight lines produced indefinitely from angles less than two right angles will meet. Therefore ΘΒ, ΖΕ, if produced, will meet. Let them be produced and let them meet at Κ, and through the point Κ let ΚΛ be drawn parallel to either ΕΑ or ΖΘ [i. 31], and let ΘΑ, ΗΒ be produced to the points Λ, Μ. Then ΘΛΚΖ is a parallelogram, ΘΚ is its diameter, and ΑΗ, ΜΕ are parallelograms, ΛΒ, ΒΖ the so-called complements, about ΘΚ. Therefore ΛΒ is equal to ΒΖ [i. 43]. But ΒΖ is equal to the triangle Γ, and therefore ΛΒ is equal to Γ [Common Notion 1]. And since the angle ΗΒΕ is equal to the angle ΑΒΜ [i. 15], while the angle ΗΒΕ is equal to Δ, therefore the angle ΑΒΜ is also equal to Δ.

Therefore the parallelogram ΛΒ, equal to the given triangle Γ, has been applied to the given straight line ΑΒ in the angle ΑΒΜ which is equal to Δ; which was to be done.[a]

γραμμον συστήσασθαι ἐν τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ (*to construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure*). The method is obvious and will not here be repeated. Proclus (*in Eucl.* i., ed. Friedlein 422. 24–423. 5, cited *infra*, p. 316) observes that it was in consequence of this problem that ancient geometers were led to investigate the squaring of the circle.

Eucl. *Elem.* ii. 5

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὸ ὑπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ἡμισείας τετραγώνῳ.

Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω εἰς μὲν ἴσα κατὰ τὸ Γ, εἰς δὲ ἄνισα κατὰ τὸ Δ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ τετραγώνῳ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΓΒ τετράγωνον τὸ ΓΕΖΒ, καὶ ἐπεζεύχθω ἡ ΒΕ, καὶ διὰ μὲν τοῦ Δ ὁποτέρᾳ τῶν ΓΕ, ΒΖ παράλληλος ἤχθω ἡ ΔΗ, διὰ δὲ τοῦ Θ ὁποτέρᾳ τῶν ΑΒ, ΕΖ παράλληλος πάλιν ἤχθω ἡ ΚΜ, καὶ πάλιν διὰ τοῦ Α ὁποτέρᾳ τῶν ΓΛ, ΒΜ παράλληλος ἤχθω ἡ ΑΚ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΓΘ παραπλήρωμα τῷ ΘΖ παραπληρώματι, κοινὸν προσκείσθω τὸ ΔΜ· ὅλον ἄρα τὸ ΓΜ ὅλῳ τῷ ΔΖ ἴσον ἐστίν. ἀλλὰ τὸ ΓΜ τῷ ΑΛ ἴσον ἐστίν, ἐπεὶ καὶ ἡ ΑΓ τῇ ΓΒ ἐστιν ἴση· καὶ τὸ ΑΛ ἄρα τῷ ΔΖ ἴσον ἐστίν. κοινὸν προσκείσθω τὸ ΓΘ· ὅλον ἄρα τὸ ΑΘ τῷ ΜΝΞ γνώμονι ἴσον ἐστίν. ἀλλὰ τὸ ΑΘ τὸ ὑπὸ τῶν ΑΔ, ΔΒ ἐστιν· ἴση γὰρ ἡ

---

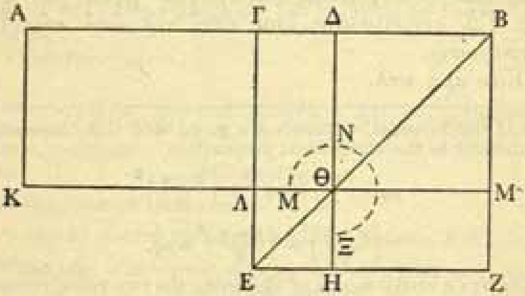
[a] Lit. "between the sections."

[b] The gnomon is indicated in the figure of the mss. by the three points M, N, Ξ and a dotted curve; there are thus in the figure two points M which should not be confused. In the next proposition a similar gnomon is described as ΝΞΟ, and perhaps this is what Euclid here wrote.

Euclid, *Elements* ii. 5

If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the line between the points of section[a] *is equal to the square on the half.*

For let a straight line ΑΒ be cut into equal segments at Γ, and into unequal segments at Δ; I say



that the rectangle contained by ΑΔ, ΔΒ with the square on ΓΔ is equal to the square on ΓΒ.

For let the square ΓΕΖΒ be described on ΓΒ [i. 46] and let ΒΕ be joined, and through Δ let ΔΗ be drawn parallel to either ΓΕ or ΒΖ, and through Θ let ΚΜ again be drawn parallel to either ΑΒ or ΕΖ, and again through Α let ΑΚ be drawn parallel to either ΓΛ or ΒΜ [i. 31]. Then, since the complement ΓΘ is equal to the complement ΘΖ [i. 43], let ΔΜ be added to each; therefore the whole ΓΜ is equal to the whole ΔΖ. But ΓΜ is equal to ΑΛ, since ΑΓ is also equal to ΓΒ [i. 36]; and therefore ΑΛ is equal to ΔΖ. Let ΓΘ be added to each; therefore the whole ΑΘ is equal to the gnomon ΜΝΞ.[b] But ΑΘ is the rect-

ΔΘ τῇ ΔΒ· καὶ ὁ ΜΝΞ ἄρα γνώμων ἴσος ἐστὶ τῷ ὑπὸ ΑΔ, ΔΒ. κοινὸν προσκείσθω τὸ ΛΗ, ὅ ἐστιν ἴσον τῷ ἀπὸ τῆς ΓΔ· ὁ ἄρα ΜΝΞ γνώμων καὶ τὸ ΛΗ ἴσα ἐστὶ τῷ ὑπὸ τῶν ΑΔ, ΔΒ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ. ἀλλὰ ὁ ΜΝΞ γνώμων καὶ τὸ ΛΗ ὅλον ἐστὶ τὸ ΓΕΖΒ τετράγωνον, ὅ ἐστιν ἀπὸ τῆς ΓΒ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ τετραγώνῳ.

Ἐὰν ἄρα κτλ.

---

[a] If the unequal segments are p, q, then this theorem is equivalent to the algebraical proposition

$$pq + \left(\frac{p+q}{2} - q\right)^2 = \left(\frac{p+q}{2}\right)^2$$

or

$$\left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2 = pq.$$

This gives a ready means of obtaining the two rules, respectively attributed to the Pythagoreans and Plato (see *supra*, pp. 90-95) for finding integral square numbers which are the sum of two other integral square numbers. Putting $p = n^2$, $q = 1$, we have

$$\left(\frac{n^2+1}{2}\right)^2 - \left(\frac{n^2-1}{2}\right)^2 = n^2.$$

In order that the first two squares may be integers, n must be odd. This is the Pythagorean rule.

Putting $$p = 2n^2,\ q = 2,$$
we have $$(n^2+1)^2 - (n^2-1)^2 = 4n^2.$$

This is Plato's rule, starting from an even number $2n$.

The theorem can be made to yield a result of even greater interest, namely, the geometrical solution of the quadratic equation

$$ax - x^2 = b^2,$$

as is shown by Heath (*The Thirteen Books of Euclid's Ele-*

angle ΑΔ, ΔΒ; for ΔΘ is equal to ΔΒ; and therefore the gnomon ΜΝΞ is equal to the rectangle ΑΔ, ΔΒ. Let ΛΗ, which is equal to the square on ΓΔ, be added to each; therefore the gnomon ΜΝΞ and ΛΗ are equal to the rectangle contained by ΑΔ, ΔΒ and the square on ΓΔ. But the gnomon ΜΝΞ and ΛΗ are the whole square ΓΕΖΒ, which is described on ΓΒ; therefore the rectangle contained by ΑΔ, ΔΒ together with the square on ΓΔ is equal to the square on ΓΒ.

Therefore, etc.[a]

ments, vol. i. p. 384, and *H.G.M.* i. 151, 152), following Simson; see also Loria, *Le scienze esatte nell' antica Grecia*, pp. 42-45.

If $$AB = a,\ \Delta B = x,$$

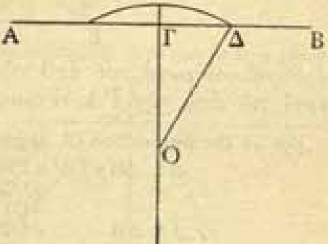
then the theorem shows that

$$(a - x) . x = \text{the rectangle } \mathrm{A}\Theta = \text{the gnomon } \mathrm{M N \Xi}.$$

If the area of the gnomon is given ($= b^2$), then we have

$$ax - x^2 = b^2.$$

To solve this equation geometrically is to find the point Δ, and in Pythagorean language this is *to apply to a given straight line* (a) *a rectangle which shall be equal to a given square* (b²) *and shall fall short by a square figure*, that is, to construct the rectangle ΑΘ or the gnomon ΜΝΞ.

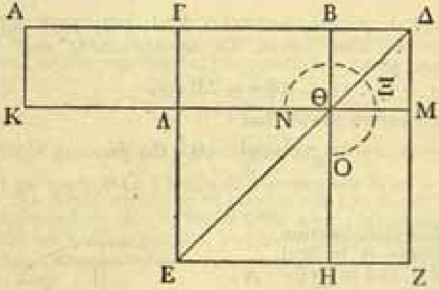


Draw ΓΟ perpendicular to AB and equal to b. With centre O and radius equal to ΓΒ ($= \frac{1}{2}a$) describe a circle. Provided that b is greater than $\frac{1}{2}a$, this circle will cut AB in two points. One of these is the required point Δ, $\Delta B = x$, and the rectangle ΑΘ can be constructed.

Eucl. *Elem.* ii. 6

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δέ τις αὐτῇ εὐθεῖα ἐπ' εὐθείας, τὸ ὑπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τῆς προσκειμένης περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς συγκειμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης τετραγώνῳ.

Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω δίχα κατὰ τὸ Γ σημεῖον, προσκείσθω δέ τις αὐτῇ εὐθεῖα ἐπ' εὐθείας ἡ ΒΔ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον



ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ.

For by the proposition (ii. 5) just proved,

$$\begin{aligned} \text{ΑΔ}\,.\,\text{ΔΒ}+\text{ΓΔ}^2 &= \text{ΓΒ}^2 \\ &= \text{ΟΔ}^2 \\ &= \text{ΟΓ}^2+\text{ΓΔ}^2 \qquad \text{(i. 47)} \\ \therefore\ \text{ΑΔ}\,.\,\text{ΔΒ} &= \text{ΟΓ}^2 \\ \text{or}\qquad (a-x)x &= b^2. \end{aligned}$$

The two points in which the circle cuts AB give two real solutions of the equation, which are coincident when $b=\frac{1}{2}a$ and the circle touches AB.

There is no direct evidence that the Pythagoreans, or

Euclid, *Elements* ii. 6

If a straight line be bisected, and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line, together with the square on the half, is equal to the square on the straight line made up of the half and the added straight line.

For let a straight line AB be bisected at the point Γ, and let a straight line BΔ be added to it in a straight line; I say that the rectangle contained by AΔ, ΔB with the square on ΓB is equal to the square on ΓΔ.[a]

Euclid for that matter, used this proposition to solve geometrically the quadratic equation $ax - x^2 = b^2$. But, as will be shown below, the Pythagoreans must have solved a similar equation corresponding to ii. 11, and it may fairly safely be assumed that they solved the equations $ax - x^2 = b^2$ corresponding to ii. 5 and the equations $ax + x^2 = b^2$ and $x^2 - ax = b^2$ corresponding to ii. 6.

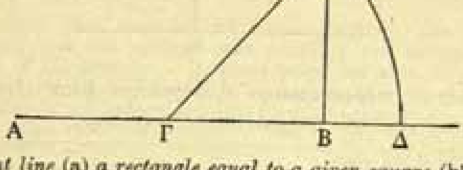
[a] The proof is on the lines of that in the preceding proposition, the rectangle AM being shown equal to the gnomon NΞO, and can easily be supplied by the reader. If $\text{AB} = a$, $\text{B}\Delta = x$, and the gnomon NΞO have a given value $(= b^2)$, then

$$(a + x) \cdot x = b^2$$

or

$$ax + x^2 = b^2.$$

To solve this equation geometrically is *to apply to a given*



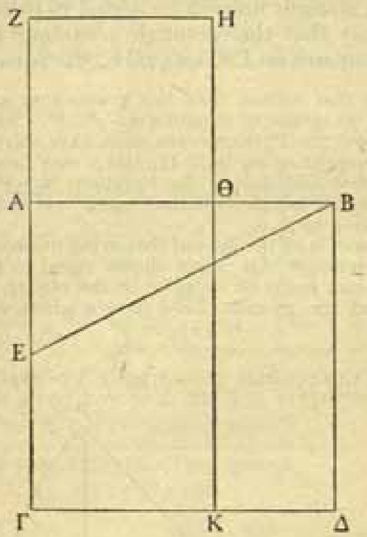
straight line (a) *a rectangle equal to a given square* (b^2) *and exceeding by a square figure*, in short, to find the point Δ.

Continued on pp. 198-199.]

Eucl. *Elem.* ii. 11

Τὴν δοθεῖσαν εὐθεῖαν τεμεῖν ὥστε τὸ ὑπὸ τῆς ὅλης καὶ τοῦ ἑτέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.

Ἔστω ἡ δοθεῖσα εὐθεῖα ἡ ΑΒ· δεῖ δὴ τὴν ΑΒ τεμεῖν ὥστε τὸ ὑπὸ τῆς ὅλης καὶ τοῦ ἑτέρου τῶν



τμημάτων περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.

Continued from p. 197.]

Simson first showed how to do this. Let BP be drawn perpendicular to AB and equal to *b*. With centre Γ and

Euclid, *Elements* ii. 11

To cut the given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

Let AB be the given straight line; then it is required to cut AB so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

---

radius ΓP let a circle be drawn cutting AB produced in Δ. Then Δ is the required point.

For by the proposition (ii. 6) just proved,

$$\begin{aligned} & A\Delta \,.\, \Delta B + \Gamma B^2 = \Gamma\Delta^2 \\ & \qquad\qquad\qquad\quad = \Gamma P^2 \\ & \qquad\qquad\qquad\quad = \Gamma B^2 + BP^2 \\ \therefore \quad & A\Delta \,.\, \Delta B \qquad\quad\; = BP^2 \\ \textit{i.e.} \quad & ax + x^2 \qquad\qquad = b^2. \end{aligned}$$

Because the circle cuts AB produced in two points there are two real solutions, and as the circle always cuts AB produced there is always a real solution. This bears out the algebraical proof that the equation

$$ax + x^2 = b^2$$

always has two real roots, which are equal when $b = \frac{1}{2}a$.

When we come to deal with Hippocrates' quadrature of lunes we shall come across the problem: To find x, when x is given by the equation

$$\sqrt{\tfrac{3}{2}}\,ax + x^2 = a^2.$$

This could have been solved theoretically by the above methods, and the solution was certainly not beyond the powers of Hippocrates. It seems more probable, however, from the wording of Eudemus's account, that he used an approximate mechanical solution for his purpose.

This same construction can be used to give a geometrical solution of the equation $x^2 - ax = b^2$. In the figure it has only to be supposed that AB $= a$ and AΔ (instead of BΔ) $= x$. Then the theorem tells us that $x(x - a) =$ the gnomon $= b^2$.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΒΔΓ, καὶ τετμήσθω ἡ ΑΓ δίχα κατὰ τὸ Ε σημεῖον, καὶ ἐπεζεύχθω ἡ ΒΕ, καὶ διήχθω ἡ ΓΑ ἐπὶ τὸ Ζ, καὶ κείσθω τῇ ΒΕ ἴση ἡ ΕΖ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΑΖ τετράγωνον τὸ ΖΘ, καὶ διήχθω ἡ ΗΘ ἐπὶ τὸ Κ· λέγω, ὅτι ἡ ΑΒ τέτμηται κατὰ τὸ Θ, ὥστε τὸ ὑπὸ τῶν ΑΒ, ΒΘ περιεχόμενον ὀρθογώνιον ἴσον ποιεῖν τῷ ἀπὸ τῆς ΑΘ τετραγώνῳ.

Ἐπεὶ γὰρ εὐθεῖα ἡ ΑΓ τέτμηται δίχα κατὰ τὸ Ε, πρόσκειται δὲ αὐτῇ ἡ ΖΑ, τὸ ἄρα ὑπὸ τῶν ΓΖ, ΖΑ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΑΕ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΖ τετραγώνῳ. ἴση δὲ ἡ ΕΖ τῇ ΕΒ· τὸ ἄρα ὑπὸ τῶν ΓΖ, ΖΑ μετὰ τοῦ ἀπὸ τῆς ΑΕ ἴσον ἐστὶ τῷ ἀπὸ ΕΒ. ἀλλὰ τῷ ἀπὸ ΕΒ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΒΑ, ΑΕ· ὀρθὴ γὰρ ἡ πρὸς τῷ Α γωνία· τὸ ἄρα ὑπὸ τῶν ΓΖ, ΖΑ μετὰ τοῦ ἀπὸ τῆς ΑΕ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΕ. κοινὸν ἀφῃρήσθω τὸ ἀπὸ τῆς ΑΕ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΓΖ, ΖΑ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ. καί ἐστι τὸ μὲν ὑπὸ τῶν ΓΖ, ΖΑ τὸ ΖΚ· ἴση γὰρ ἡ ΑΖ τῇ ΖΗ· τὸ δὲ ἀπὸ τῆς ΑΒ τὸ ΑΔ· τὸ ἄρα ΖΚ ἴσον ἐστὶ τῷ ΑΔ. κοινὸν ἀφῃρήσθω τὸ ΑΚ· λοιπὸν ἄρα τὸ ΖΘ τῷ ΘΔ ἴσον ἐστίν. καί ἐστι τὸ μὲν ΘΔ τὸ ὑπὸ τῶν ΑΒ, ΒΘ· ἴση γὰρ ἡ ΑΒ τῇ ΒΔ· τὸ δὲ ΖΘ τὸ ἀπὸ τῆς ΑΘ· τὸ ἄρα ὑπὸ τῶν ΑΒ, ΒΘ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ ΘΑ τετραγώνῳ.

Ἡ ἄρα κτλ.

---

[a] If $AB = a$, $A\Theta = x$, then AB has been so cut at Θ that

$$a(a-x) = x^2$$

or

$$x^2 + ax = a^2.$$

Let the square ΑΒΔΓ be described on ΑΒ, and let ΑΓ be bisected at the point Ε, and let ΒΕ be joined, and let ΓΑ be produced to Ζ, and let ΕΖ be made equal to ΒΕ, and let the square ΖΘ be described on ΑΖ, and let ΗΘ be produced to Κ; I say that ΑΒ has been so cut at Θ as to make the rectangle contained by ΑΒ, ΒΘ equal to the square on ΑΘ.

For, since the straight line ΑΓ has been bisected at Ε, and ΖΑ is added to it, therefore the rectangle contained by ΓΖ, ΖΑ together with the square on ΑΕ is equal to the square on ΕΖ [ii. 6]. But ΕΖ is equal to ΕΒ; therefore the rectangle contained by ΓΖ, ΖΑ together with the square on ΑΕ is equal to the square on ΕΒ. But the squares on ΒΑ, ΑΕ are equal to the square on ΕΒ, for the angle at Α is right [i. 47]; therefore the rectangle contained by ΓΖ, ΖΑ together with the square on ΑΕ is equal to the squares on ΒΑ, ΑΕ. Let the square on ΑΕ be taken away from each; therefore the rectangle contained by ΓΖ, ΖΑ which remains is equal to the square on ΑΒ. Now the rectangle ΓΖ, ΖΑ is ΖΚ, for ΑΖ is equal to ΖΗ; and the square on ΑΒ is ΑΔ; therefore ΖΚ is equal to ΑΔ. Let ΑΚ be taken away from each; therefore the remainder ΖΘ is equal to ΘΔ. Now ΘΔ is the rectangle ΑΒ, ΒΘ, for ΑΒ is equal to ΒΔ; and ΖΘ is the square on ΑΘ; therefore the rectangle contained by ΑΒ, ΒΘ is equal to the square on ΘΑ.

Therefore, etc.[a]

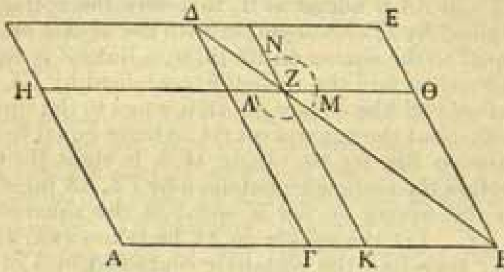
[a] In other words, the proposition gives a geometrical solution of the equation
$$x^2 + ax = a^2$$
for it enables us to find ΑΘ or x.

This equation is a particular case of the more general proposition
$$x^2 + ax = b^2$$
which, as was explained in the note on p. 197 n. a, can be solved

Eucl. *Elem.* vi. 27

Πάντων τῶν παρὰ τὴν αὐτὴν εὐθεῖαν παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἴδεσι παραλληλογράμμοις ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ἀπὸ τῆς ἡμισείας ἀναγραφομένῳ μέγιστόν ἐστι τὸ ἀπὸ τῆς ἡμισείας παραβαλλόμενον ὅμοιον ὂν τῷ ἐλλείματι.

Ἔστω εὐθεῖα ἡ ΑΒ καὶ τετμήσθω δίχα κατὰ



by a method based on ii. 6. There is good reason to believe, as will be shown below, pp. 222-225, that the Pythagoreans knew how to construct a regular pentagon ABCDE, and it is probable that this theorem was used in the construction, as can be shown if CE is allowed to cut AD in F.

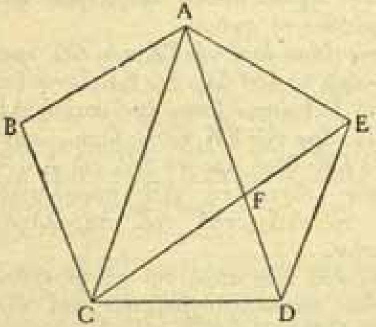
For the Pythagoreans, knowing that the sum of the angles of any triangle is two right angles, would immediately have deduced that the sum of the internal angles of a regular pentagon is six right angles, and that each of the internal angles is therefore $\frac{6}{5}$ths of a right angle. It easily follows that the angles CAD, ADC, DCA are respectively $\frac{2}{5}$ths, $\frac{6}{5}$ths and $\frac{2}{5}$ths of a right angle, while the angles FCD, CDF, DFC are also respectively $\frac{2}{5}$ths, $\frac{2}{5}$ths and $\frac{6}{5}$ths of a right angle. From this it follows that the triangles ACD, CDF are similar, while AF = FC = CD.

Euclid, *Elements* vi. 27

Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect.[a]

Let AB be a straight line and let it be bisected

Therefore	AC : CD = CD : DF
or	AD : AF = AF : FD
or	AD . FD = AF².



The point F can therefore be found according to the method of ii. 6, and the pentagon constructed, starting from AD.

[a] This proposition gives the conditions under which it is possible to solve the next proposition, and so full consideration will be left to the note on p. 210. It is the first example we have met of a διορισμός. It will be remembered that according to Proclus Leon discovered διορισμοί (see *supra*, p. 150).

τὸ Γ, καὶ παραβεβλήσθω παρὰ τὴν ΑΒ εὐθεῖαν τὸ ΑΔ παραλληλόγραμμον ἐλλεῖπον εἴδει παραλληλογράμμῳ τῷ ΔΒ ἀναγραφέντι ἀπὸ τῆς ἡμισείας τῆς ΑΒ, τουτέστι τῆς ΓΒ· λέγω, ὅτι πάντων τῶν παρὰ τὴν ΑΒ παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἴδεσι ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ΔΒ μέγιστόν ἐστι τὸ ΑΔ. παραβεβλήσθω γὰρ παρὰ τὴν ΑΒ εὐθεῖαν τὸ ΑΖ παραλληλόγραμμον ἐλλεῖπον εἴδει παραλληλογράμμῳ τῷ ΖΒ ὁμοίῳ τε καὶ ὁμοίως κειμένῳ τῷ ΔΒ· λέγω, ὅτι μεῖζόν ἐστι τὸ ΑΔ τοῦ ΑΖ.

Ἐπεὶ γὰρ ὅμοιόν ἐστι τὸ ΔΒ παραλληλόγραμμον τῷ ΖΒ παραλληλογράμμῳ, περὶ τὴν αὐτήν εἰσι διάμετρον. ἤχθω αὐτῶν διάμετρος ἡ ΔΒ, καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΓΖ τῷ ΖΕ, κοινὸν δὲ τὸ ΖΒ, ὅλον ἄρα τὸ ΓΘ ὅλῳ τῷ ΚΕ ἐστιν ἴσον. ἀλλὰ τὸ ΓΘ τῷ ΓΗ ἐστιν ἴσον, ἐπεὶ καὶ ἡ ΑΓ τῇ ΓΒ. καὶ τὸ ΗΓ ἄρα τῷ ΕΚ ἐστιν ἴσον. κοινὸν προσκείσθω τὸ ΓΖ· ὅλον ἄρα τὸ ΑΖ τῷ ΛΜΝ γνώμονί ἐστιν ἴσον· ὥστε τὸ ΔΒ παραλληλόγραμμον, τουτέστι τὸ ΑΔ, τοῦ ΑΖ παραλληλογράμμου μεῖζόν ἐστιν.

Πάντων ἄρα τῶν παρὰ τὴν αὐτὴν εὐθεῖαν παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἴδεσι παραλληλογράμμοις ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ἀπὸ τῆς ἡμισείας ἀναγραφομένῳ μέγιστόν ἐστι τὸ ἀπὸ τῆς ἡμισείας παραβληθέν· ὅπερ ἔδει δεῖξαι.

Eucl. *Elem.* vi. 28

Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι εὐθυ-

at Γ, and let there be applied to the straight line AB the parallelogram AΔ deficient by the parallelogrammic figure ΔB described on the half of AB, that is, ΓB. I say that, of all the parallelograms applied to AB and deficient by figures similar and similarly situated to ΔB, AΔ is the greatest. For let there be applied to the straight line AB the parallelogram AZ deficient by the parallelogrammic figure ZB similar and similarly situated to ΔB. I say that AΔ is greater than AZ.

For since the parallelogram ΔB is similar to the parallelogram ZB, they are about the same diameter. Let their diameter ΔB be drawn and let the figure be described.

Then, since ΓZ is equal to ZE, and ZB is common, the whole ΓΘ is equal to the whole KE. But ΓΘ is equal to ΓH, since AΓ is equal to ΓB. And therefore HΓ is equal to EK. Let ΓZ be added to each. Then the whole AZ is equal to the gnomon ΛMN, so that the parallelogram ΔB, that is, AΔ, is greater than the parallelogram AZ.

Therefore of all the parallelograms applied to this straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line the greatest is that applied from the half; which was to be proved.

Euclid, *Elements* vi. 28

To the given straight line to apply a parallelogram

γράμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ἐλλεῖπον εἴδει παραλληλογράμμῳ ὁμοίῳ τῷ δοθέντι· δεῖ δὲ τὸ διδόμενον εὐθύγραμμον [ᾧ δεῖ ἴσον παραβαλεῖν][1] μὴ μεῖζον εἶναι τοῦ ἀπὸ τῆς ἡμισείας ἀναγραφομένου ὁμοίου τῷ ἐλλείματι [τοῦ τε ἀπὸ τῆς ἡμισείας καὶ ᾧ δεῖ ὅμοιον ἐλλείπειν].[1]

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν εὐθύγραμμον, ᾧ δεῖ ἴσον παρὰ τὴν ΑΒ παραβαλεῖν τὸ Γ μὴ μεῖζον [ὂν] τοῦ ἀπὸ τῆς ἡμισείας τῆς ΑΒ ἀναγραφομένου ὁμοίου τῷ ἐλλείμματι, ᾧ δὲ δεῖ ὅμοιον ἐλλείπειν, τὸ Δ· δεῖ δὴ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβαλεῖν ἐλλεῖπον εἴδει παραλληλογράμμῳ ὁμοίῳ ὄντι τῷ Δ.

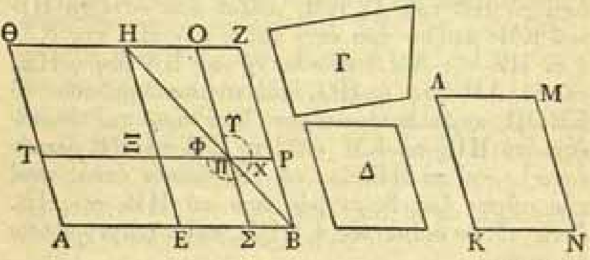
Τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ε σημεῖον, καὶ ἀναγεγράφθω ἀπὸ τῆς ΕΒ τῷ Δ ὅμοιον καὶ ὁμοίως κείμενον τὸ ΕΒΖΗ, καὶ συμπεπληρώσθω τὸ ΑΗ παραλληλόγραμμον.

Εἰ μὲν οὖν ἴσον ἐστὶ τὸ ΑΗ τῷ Γ, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν· παραβέβληται γὰρ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον τὸ ΑΗ ἐλλεῖπον εἴδει παραλληλογράμμῳ τῷ ΗΒ ὁμοίῳ ὄντι τῷ Δ. εἰ δὲ οὔ,

[1] The bracketed words are interpolations by Theon in his recension of the *Elements* (Heiberg).

equal to the given rectilineal figure and deficient by a parallelogrammic figure similar to the given one; thus the given rectilineal figure must be not greater than the [parallelogram] described on the half [of the straight line] and similar to the defect.

Let AB be the given straight line, Γ the given rectilineal figure, to which the figure to be applied



to AB is required to be equal, being not greater than the [parallelogram] described on the half [of the straight line] and similar to the defect, and Δ the [parallelogram] to which the defect is required to be similar; then it is required to apply to the given straight line AB a parallelogram equal to the given rectilineal figure Γ and deficient by a parallelogrammic form similar to Δ.

Let AB be bisected at the point E, and on E let EBZH be described similar and similarly situated to Δ [vi. 18], and let the parallelogram AH be completed.

If then AH is equal to Γ, that which was enjoined will have been done; for there has been applied to the given straight line AB a parallelogram AH equal to the given rectilineal figure Γ and deficient by a parallelogrammic figure HB similar to Δ. But if not,

μεῖζον ἔστω τὸ ΘΕ τοῦ Γ. ἴσον δὲ τὸ ΘΕ τῷ ΗΒ· μεῖζον ἄρα καὶ τὸ ΗΒ τοῦ Γ. ᾧ δὴ μεῖζόν ἐστι τὸ ΗΒ τοῦ Γ, ταύτῃ τῇ ὑπεροχῇ ἴσον, τῷ δὲ Δ ὅμοιον καὶ ὁμοίως κείμενον τὸ αὐτὸ συνεστάτω τὸ ΚΛΜΝ. ἀλλὰ τὸ Δ τῷ ΗΒ [ἐστιν] ὅμοιον· καὶ τὸ ΚΜ ἄρα τῷ ΗΒ ἐστιν ὅμοιον. ἔστω οὖν ὁμόλογος ἡ μὲν ΚΛ τῇ ΗΕ, ἡ δὲ ΛΜ τῇ ΗΖ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΗΒ τοῖς Γ, ΚΜ, μεῖζον ἄρα ἐστὶ τὸ ΗΒ τοῦ ΚΜ· μείζων ἄρα ἐστὶ καὶ ἡ μὲν ΗΕ τῆς ΚΛ, ἡ δὲ ΗΖ τῆς ΛΜ. κείσθω τῇ μὲν ΚΛ ἴση ἡ ΗΞ, τῇ δὲ ΛΜ ἴση ἡ ΗΟ, καὶ συμπεπληρώσθω τὸ ΞΗΟΠ παραλληλόγραμμον· ἴσον ἄρα καὶ ὅμοιόν ἐστι [τὸ ΗΠ] τῷ ΚΜ [ἀλλὰ τὸ ΚΜ τῷ ΗΒ ὅμοιόν ἐστιν]. καὶ τὸ ΗΠ ἄρα τῷ ΗΒ ὅμοιόν ἐστιν· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὸ ΗΠ τῷ ΗΒ. ἔστω αὐτῶν διάμετρος ἡ ΗΠΒ, καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΒΗ τοῖς Γ, ΚΜ, ὧν τὸ ΗΠ τῷ ΚΜ ἐστιν ἴσον, λοιπὸς ἄρα ὁ ΥΧΦ γνώμων λοιπῷ τῷ Γ ἴσος ἐστίν. καὶ ἐπεὶ ἴσον ἐστὶ τὸ ΟΡ τῷ ΞΣ, κοινὸν προσκείσθω τὸ ΠΒ· ὅλον ἄρα τὸ ΟΒ ὅλῳ τῷ ΞΒ ἴσον ἐστίν. ἀλλὰ τὸ ΞΒ τῷ ΤΕ ἐστιν ἴσον, ἐπεὶ καὶ πλευρὰ ἡ ΑΕ πλευρᾷ τῇ ΕΒ ἐστιν ἴση· καὶ τὸ ΤΕ ἄρα τῷ ΟΒ ἐστιν ἴσον. κοινὸν προσκείσθω τὸ ΞΣ· ὅλον ἄρα τὸ ΤΣ ὅλῳ τῷ ΦΧΥ γνώμονί ἐστιν ἴσον. ἀλλ' ὁ ΦΧΥ γνώμων τῷ Γ ἐδείχθη ἴσος· καὶ τὸ ΤΣ ἄρα τῷ Γ ἐστιν ἴσον.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΣΤ ἐλλεῖπον εἴδει παραλληλογράμμῳ τῷ ΠΒ ὁμοίῳ ὄντι τῷ Δ [ἐπειδή-

let ΘE be greater than Γ. Now ΘE is equal to HB and therefore HB is greater than Γ. Let KΛMN be constructed at once equal to the excess by which HB is greater than Γ and similar and similarly situated to Δ [vi. 25]. But Δ is similar to HB; therefore KM is also similar to HB [vi. 21]. Let KΛ correspond to HE, ΛM to HZ. Now, since HB is equal to Γ + KM, HB is therefore greater than KM. Therefore HE is greater than KΛ, and HZ than ΛM. Let HΞ be made equal to KΛ, and HO equal to ΛM, and let the parallelogram ΞHOΠ be completed. Therefore it is equal and similar to KM. Therefore HΠ is also similar to HB. Therefore HΠ is about the same diameter as HB [vi. 26]. Let HΠB be their diameter, and let the figure be described.

Then since BH is equal to Γ + KM, and in these HΠ is equal to KM, therefore the remainder, the gnomon ΥXΦ, is equal to Γ. And since OP is equal to ΞΣ, let ΠB be added to each. Therefore the whole of OB is equal to the whole of ΞB. But ΞB is equal to TE, since the side AE is also equal to the side EB [i. 36]. Therefore TE is also equal to OB. Let ΞΣ be added to both. Therefore the whole of TΣ is equal to the whole of the gnomon ΦXΥ. But the gnomon ΦXΥ was proved equal to Γ. Therefore TΣ is also equal to Γ.

Therefore to the given straight line AB there has been applied the parallelogram ΣT equal to the given rectilineal figure Γ and deficient by a parallelo-

περ τὸ ΠΒ τῷ ΗΠ ὅμοιόν ἐστιν]· ὅπερ ἔδει ποιῆσαι.

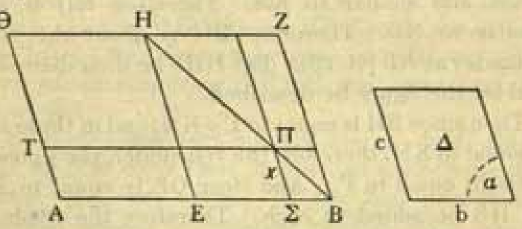
Eucl. *Elem.* vi. 29

Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι εὐθυγράμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ὑπερβάλλον εἴδει παραλληλογράμμῳ ὁμοίῳ τῷ δοθέντι.

Ἔστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν εὐθύγραμμον, ᾧ δεῖ ἴσον παρὰ τὴν ΑΒ παραβαλεῖν,

---

[a] If AB $= a$, ΣΠ $= x$, while the sides of the given parallelogram Δ are in the ratio $b : c$, and the angle of Δ is α, then ΣΒ $= \frac{b}{c}x$, and



(the parallelogram ΤΣ) = (the parallelogram ΤΒ) − (the parallelogram ΠΒ)

$$= ax \sin \alpha - \frac{b}{c}x \, . \, x \sin \alpha.$$

If the area of the given rectilineal figure Γ is S, the proposition tells us that

$$ax \sin \alpha - \frac{b}{c}x^2 \sin \alpha = S.$$

To construct the parallelogram ΤΣ is therefore equivalent to solving geometrically the equation

$$ax - \frac{b}{c}x^2 = \frac{S}{\sin \alpha}.$$

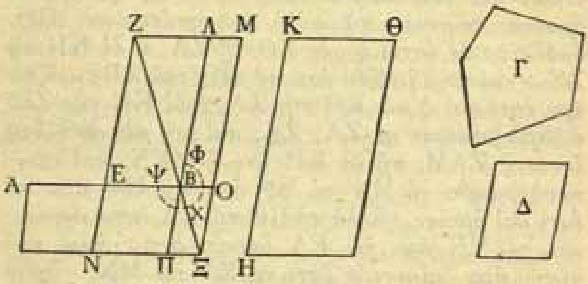
Heath (*The Thirteen Books of Euclid's Elements*, vol. ii.

grammic form ΠΒ similar to Δ; which was to be done.[a]

Euclid, *Elements* vi. 29

To the given straight line to apply a parallelogram equal to the given rectilineal figure and exceeding by a parallelogrammic figure similar to the given one.

Let AB be the given straight line, Γ the given



rectilineal figure to which the figure to be applied to

pp. 263-264), shows how the geometrical method is precisely equivalent to the algebraical method of completing the square on the left-hand side, and he demonstrates how the *two* solutions can be obtained geometrically, though Euclid, consistently with his practice, gives one only.

For a real solution it is necessary, as every schoolboy knows, that

$$\frac{S}{\sin \alpha} \nless \frac{c}{b} \cdot \frac{a^2}{4}$$

$$\text{i.e.}\quad S \nless \left(\frac{c}{b} \cdot \frac{a}{2}\right) (\sin \alpha) \left(\frac{a}{2}\right)$$

$$\text{i.e.}\quad S \nless \text{HE} \sin \alpha \,.\, \text{EB}$$

$$\text{i.e.}\quad S \nless \text{parallelogram HB.}$$

This is precisely the result obtained in vi. 27.

τὸ Γ, ᾧ δὲ δεῖ ὅμοιον ὑπερβάλλειν, τὸ Δ· δεῖ δὴ παρὰ τὴν ΑΒ εὐθεῖαν τῷ Γ εὐθυγράμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ὑπερβάλλον εἴδει παραλληλογράμμῳ ὁμοίῳ τῷ Δ.

Τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ε, καὶ ἀναγεγράφθω ἀπὸ τῆς ΕΒ τῷ Δ ὅμοιον καὶ ὁμοίως κείμενον παραλληλόγραμμον τὸ ΒΖ, καὶ συναμφοτέροις μὲν τοῖς ΒΖ, Γ ἴσον, τῷ δὲ Δ ὅμοιον καὶ ὁμοίως κείμενον τὸ αὐτὸ συνεστάτω τὸ ΗΘ. ὁμόλογος δὲ ἔστω ἡ μὲν ΚΘ τῇ ΖΛ, ἡ δὲ ΚΗ τῇ ΖΕ. καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΗΘ τοῦ ΖΒ, μείζων ἄρα ἐστὶ καὶ ἡ μὲν ΚΘ τῆς ΖΛ, ἡ δὲ ΚΗ τῆς ΖΕ. ἐκβεβλήσθωσαν αἱ ΖΛ, ΖΕ, καὶ τῇ μὲν ΚΘ ἴση ἔστω ἡ ΖΛΜ, τῇ δὲ ΚΗ ἴση ἡ ΖΕΝ, καὶ συμπεπληρώσθω τὸ ΜΝ· τὸ ΜΝ ἄρα τῷ ΗΘ ἴσον τέ ἐστι καὶ ὅμοιον. ἀλλὰ τὸ ΗΘ τῷ ΕΛ ἐστιν ὅμοιον· καὶ τὸ ΜΝ ἄρα τῷ ΕΛ ὅμοιόν ἐστιν· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὸ ΕΛ τῷ ΜΝ. ἤχθω αὐτῶν διάμετρος ἡ ΖΞ, καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ ἴσον ἐστὶ τὸ ΗΘ τοῖς ΕΛ, Γ, ἀλλὰ τὸ ΗΘ τῷ ΜΝ ἴσον ἐστίν, καὶ τὸ ΜΝ ἄρα τοῖς ΕΛ, Γ ἴσον ἐστίν. κοινὸν ἀφῃρήσθω τὸ ΕΛ· λοιπὸς ἄρα ὁ ΨΧΦ γνώμων τῷ Γ ἐστιν ἴσος. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῇ ΕΒ, ἴσον ἐστὶ καὶ τὸ ΑΝ τῷ ΝΒ, τουτέστι τῷ ΛΟ. κοινὸν προσκείσθω τὸ ΕΞ· ὅλον ἄρα τὸ ΑΞ ἴσον ἐστὶ τῷ ΦΧΨ γνώμονι. ἀλλὰ ὁ ΦΧΨ γνώμων τῷ Γ ἴσος ἐστίν· καὶ τὸ ΑΞ ἄρα τῷ Γ ἴσον ἐστίν.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΑΞ ὑπερβάλλον εἴδει παραλληλο-

AB is required to be equal, and Δ that to which the excess is required to be similar; then it is required to apply to the straight line AB a parallelogram equal to the rectilineal figure Γ and exceeding by a parallelogrammic figure similar to Δ.

Let AB be bisected at E, and let there be described on EB the parallelogram BZ similar and similarly situated to Δ, and let HΘ be constructed at once equal to the sum of BZ, Γ and similar and similarly situated to Δ. Let KΘ correspond to ZΛ and KH to ZE. Now since HΘ is greater than ZB, KΘ is therefore greater than ZΛ, and KH than ZE. Let ZΛ, ZE be produced, and let ZΛM be equal to KΘ, and ZEN equal to KH, and let MN be completed; therefore MN is both equal to HΘ and similar. But HΘ is similar to EΛ; therefore MN is similar to EΛ [vi. 21]; and therefore EΛ is about the same diameter with MN [vi. 26]. Let their diameter ZΞ be drawn, and let the figure be described.

Since HΘ is equal to EΛ + Γ, while HΘ is equal to MN, therefore MN is also equal to EΛ + Γ. Let EΛ be taken away from each; therefore the remainder, the gnomon ΨXΦ, is equal to Γ. And since AE is equal to EB, AN is also equal to NB [i. 36], that is, to ΛO [i. 43]. Let EΞ be added to each; therefore the whole of AΞ is equal to the gnomon ΦXΨ. But the gnomon ΦXΨ is equal to Γ; therefore AΞ is also equal to Γ.

Therefore to the given straight line AB there has been applied a parallelogram AΞ equal to the given rectilineal figure Γ and exceeding by a parallelo-

γράμμῳ τῷ ΠΟ ὁμοίῳ ὄντι τῷ Δ, ἐπεὶ καὶ τῷ ΕΛ ἐστιν ὅμοιον τὸ ΟΠ· ὅπερ ἔδει ποιῆσαι.

(e) The Irrational

Schol. i. in Eucl. *Elem.* x., Eucl. ed. Heiberg v. 415. 7–417. 14

Ἦλθον δὲ τὴν ἀρχὴν ἐπὶ τὴν τῆς συμμετρίας ζήτησιν οἱ Πυθαγόρειοι πρῶτοι αὐτὴν ἐξευρόντες ἐκ τῆς τῶν ἀριθμῶν κατανοήσεως. κοινοῦ γὰρ ἁπάντων ὄντος μέτρου τῆς μονάδος καὶ ἐπὶ τῶν μεγεθῶν κοινὸν μέτρον εὑρεῖν οὐκ ἠδυνήθησαν. αἴτιον δὲ τὸ πάντα μὲν καὶ ὁποιονοῦν ἀριθμὸν καθ' ὁποιασοῦν τομὰς διαιρούμενον μόριόν τι καταλιμπάνειν ἐλάχιστον καὶ τομῆς ἀνεπίδεκτον, πᾶν δὲ μέγεθος ἐπ' ἄπειρον διαιρούμενον μὴ καταλιμπάνειν μόριον, ὃ διὰ τὸ εἶναι ἐλάχιστον τομὴν οὐκ ἐπιδέξεται, ἀλλὰ καὶ ἐκεῖνο ἐπ' ἄπειρον τεμνόμενον ποιεῖν ἄπειρα μόρια, ὧν ἕκαστον ἐπ' ἄπειρον τμηθήσεται, καὶ ἁπλῶς τὸ μὲν μέγεθος κατὰ μὲν τὸ μερίζεσθαι μετέχειν τῆς τοῦ ἀπείρου ἀρχῆς, κατὰ δὲ τὴν ὁλότητα τῆς τοῦ πέρατος, τὸν δὲ ἀριθμὸν κατὰ μὲν τὸ μερίζεσθαι τῆς τοῦ πέρατος,

---

[a] If the angle of Δ is α and its sides are in the ratio $b:c$, while AB $=a$ and ΟΞ $=x$, then

$$\text{(parallelogram ΑΞ)} = \text{(parallelogram ΑΠ)} + \text{(parallelogram ΒΞ)}$$
$$= ax\sin\alpha + \frac{b}{c}x \cdot x\sin\alpha.$$

grammic form ΠΟ similar to Δ, since ΟΠ is similar to ΕΛ; which was to be done.[a]

(e) The Irrational [b]

Euclid, *Elements* x., Scholium i., Eucl. ed. Heiberg v. 415. 7–417. 14

The Pythagoreans were the first to make inquiry into commensurability, having first discovered it as a result of their observation of numbers; for though the unit is a common measure of all numbers they could not find a common measure of all magnitudes. The reason is that all numbers, of whatsoever kind, howsoever they be divided leave some least part which will not suffer further division; but all magnitudes are divisible *ad infinitum* and do not leave some part which, being the least possible, will not admit of further division, but that remainder can be divided *ad infinitum* so as to give an infinite number of parts, of which each can be divided *ad infinitum*; and, in sum, magnitude partakes in division of the principle of the infinite, but in its entirety of the principle of the finite, while number in division partakes of the

But by the proposition, if S is the area of Γ

$$(\text{parallelogram } \mathrm{A}\Xi) = \mathrm{S},$$

$$\therefore \quad ax + \frac{b}{c}x^2 = \frac{\mathrm{S}}{\sin \alpha}.$$

To construct the parallelogram ΑΞ is therefore equivalent to solving geometrically this quadratic equation. There is always a real solution, and so no διορισμός is necessary as in the case of the preceding proposition. Heath (*The Thirteen Books of Euclid's Elements*, vol. ii. pp. 266-267) again shows how the procedure is equivalent to the algebraic method of completing the square. Euclid's solution corresponds to the root with the positive sign.

[b] For further notices see *supra*, pp. 110-111, p. 149 n. c.

κατὰ δὲ τὴν ὁλότητα τῆς τοῦ ἀπείρου . . . τῶν γὰρ Πυθαγορείων λόγος τὸν πρῶτον τὴν περὶ τούτων θεωρίαν εἰς τοὐμφανὲς ἐξαγαγόντα ναυαγίῳ περιπεσεῖν.

(*f*) The Five Regular Solids

Phil. ap. Stob. *Ecl.* 1, proem. 3, ed. Wachsmuth 18. 5; Diels, *Vors.* i[3]. 412. 15–413. 2

Καὶ τὰ μὲν τᾶς σφαίρας σώματα πέντε ἐντί, τὰ ἐν τᾷ σφαίρᾳ πῦρ ⟨καὶ⟩ ὕδωρ καὶ γᾶ καὶ ἀήρ, καὶ ὁ τᾶς σφαίρας ὁλκάς,[1] πέμπτον.

Aët. *Plac.* ii. 6. 5; Diels, *Vors.* i[3]. 403. 8-12

Πυθαγόρας πέντε σχημάτων ὄντων στερεῶν, ἅπερ καλεῖται καὶ μαθηματικά, ἐκ μὲν τοῦ κύβου φησὶ γεγονέναι τὴν γῆν, ἐκ δὲ τῆς πυραμίδος τὸ πῦρ, ἐκ δὲ τοῦ ὀκταέδρου τὸν ἀέρα, ἐκ δὲ τοῦ εἰκοσαέδρου τὸ ὕδωρ, ἐκ δὲ τοῦ δωδεκαέδρου τὴν τοῦ παντὸς σφαῖραν.

[1] ὁλκάς : ὁλκός coniecit Wilamowitz.

---

[a] A regular solid is one having all its faces equal polygons and all its solid angles equal. The term is usually restricted to those regular solids in which the centre is singly enclosed. There are five, and only five, such figures—the pyramid, cube, octahedron, dodecahedron and icosahedron. They can all be inscribed in a sphere. Owing to the use made of them in Plato's *Timaeus* for the construction of the universe they were often called by the Greeks the *cosmic* or *Platonic* figures. As noted above (p. 148), Proclus attributes the construction of the cosmic figures to Pythagoras, but Suidas (*infra*, p. 378) says Theaetetus was the first to write on them. The theoretical construction of the regular solids and the calculation of their sides in terms of the radius of the circumscribed sphere occupies Book xiii. of Euclid's *Elements*. It

finite, but in its entirety of the infinite. . . . There is a legend that the first of the Pythagoreans who made public the investigation of these matters perished in a shipwreck.

(*f*) The Five Regular Solids[a]

Philolaus, cited by Stobaeus, *Extracts* i. proœm. 3, ed. Wachsmuth 18. 5 ; Diels, *Vors.* i[3]. 412. 15–413. 2

There are five bodies pertaining to the sphere—the fire, water, earth and air in the sphere, and the vessel of the sphere itself as the fifth.[b]

Aëtius, *Placita* ii. 6. 5 ; Diels, *Vors.* i[3]. 403. 8-12

Pythagoras, seeing that there are five solid figures, which are also called the mathematical figures, says that the earth arose from the cube, fire from the pyramid, air from the octahedron, water from the icosahedron, and the sphere of the universe from the dodecahedron.[c]

calls for mathematical knowledge which the Pythagoreans did not possess ; but there is no reason why the Pythagoreans should not have constructed them practically in the manner of Plato by putting together triangles, squares or pentagons. The passages here given almost compel that conclusion.

The subject is fully treated in *Die fünf Platonischen Körper*, by Eva Sachs (*Philologische Untersuchungen*, 21es Heft, 1917). Archimedes, according to Pappus, *Coll.* v., ed. Hultsch 352-358, discovered thirteen semi-regular solids, whose faces are all regular polygons, but not all of the same kind.

[b] In place of ὁλκάς Wilamowitz suggests ὁλκός, which is derived from ἕλκω and could be translated " envelope." This fragment, it will be noted, does not identify the regular solids with the elements in the sphere, but it is consistent with that identification, for which the earliest definite evidence is Plato's *Timaeus*.

[c] Aëtius's authority is probably Theophrastus.

Plat. *Tim.* 53 c–55 c

Πρῶτον μὲν δὴ πῦρ καὶ γῆ καὶ ὕδωρ καὶ ἀὴρ ὅτι σώματά ἐστι, δῆλόν που καὶ παντί. τὸ δὲ τοῦ σώματος εἶδος πᾶν καὶ βάθος ἔχει. τὸ δὲ βάθος αὖ πᾶσα ἀνάγκη τὴν ἐπίπεδον περιειληφέναι φύσιν. ἡ δὲ ὀρθὴ τῆς ἐπιπέδου βάσεως ἐκ τριγώνων συνέστηκε. τὰ δὲ τρίγωνα πάντα ἐκ δυοῖν ἄρχεται τριγώνοιν, μίαν μὲν ὀρθὴν ἔχοντος ἑκατέρου γωνίαν, τὰς δὲ ὀξείας· ὧν τὸ μὲν ἕτερον ἑκατέρωθεν ἔχει μέρος γωνίας ὀρθῆς πλευραῖς ἴσαις διῃρημένης, τὸ δὲ ἕτερον ἀνίσοις ἄνισα μέρη νενεμημένης. . . .

Τοῖν δὴ δυοῖν τριγώνοιν τὸ μὲν ἰσοσκελὲς μίαν εἴληχε φύσιν, τὸ δὲ πρόμηκες ἀπεράντους. προαιρετέον οὖν αὖ τῶν ἀπείρων τὸ κάλλιστον, εἰ μέλλομεν ἄρξεσθαι κατὰ τρόπον. ἂν οὖν τις ἔχῃ κάλλιον ἐκλεξάμενος εἰπεῖν εἰς τὴν τούτων σύστασιν, ἐκεῖνος οὐκ ἐχθρὸς ὢν ἀλλὰ φίλος κρατεῖ· τιθέμεθα δ' οὖν τῶν πολλῶν τριγώνων κάλλιστον ἕν, ὑπερβάντες τἆλλα, ἐξ οὗ τὸ ἰσόπλευρον τρίγωνον ἐκ τρίτου συνέστηκεν. . . .

Οἷον δὲ ἕκαστον αὐτῶν γέγονεν εἶδος καὶ ἐξ ὅσων συμπεσόντων ἀριθμῶν, λέγειν ἂν ἑπόμενον εἴη. ἄρξει δὴ τό τε πρῶτον εἶδος καὶ σμικρότατον συνιστάμενον· στοιχεῖον δ' αὐτοῦ τὸ τὴν ὑποτείνουσαν τῆς ἐλάττονος πλευρᾶς διπλασίαν ἔχον μήκει· σύνδυο δὲ τοιούτων κατὰ διάμετρον συντιθεμένων καὶ τρὶς τούτου γενομένου, τὰς διαμέτρους

---

[a] This passage is put into the mouth of Timaeus of Locri, a Pythagorean leader, and in it Plato is generally held to be reproducing Pythagorean ideas.

[b] *i.e.*, the rectangular isosceles triangle and the rectangular scalene triangle.

Plato, *Timaeus* 53 c–55 c [a]

In the first place, then, it is clear to everyone, I think, that fire and earth and water and air are bodies. Now in every case the form of a body has depth. Further, it is absolutely necessary that depth should be bounded by a plane surface; and the rectilinear plane is composed of triangles. Now all triangles have their origin in two triangles, each having one right angle and the others acute; and one of these triangles has on each side half a right angle marked off by equal sides, while the other has the right angle divided into unequal parts by unequal sides.[b] . . .

Of the two triangles, the isosceles has one nature only, but the scalene has an infinite number; and of these infinite natures the fairest must be chosen, if we would make a suitable beginning. If, then, anyone can claim that he has a fairer one for the construction of these bodies, he is no foe but shall prevail as a friend; but we shall pass over all the rest and lay down as the fairest of the many triangles that from which the equilateral triangle arises as a third when two are conjoined. . . .[c]

In the next place we have to describe the form in which each kind has come into existence and from what numbers it is compounded. A beginning must be made with that kind which is primary and has the smallest components, and its element is the triangle whose hypotenuse is twice as long as the lesser side. When a pair of these triangles are joined diagonally and this is done three times, by drawing the hypo-

[c] *i.e.*, the "fairest" of rectangular scalene triangles is half of an equilateral triangle, the sides being in the proportion 1, $\sqrt{3}$, 2.

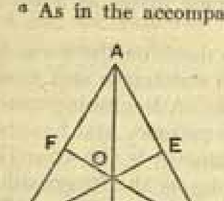
καὶ τὰς βραχείας πλευρὰς εἰς ταὐτὸν ὡς κέντρον ἐρεισάντων, ἓν ἰσόπλευρον τρίγωνον ἐξ ἓξ τὸν ἀριθμὸν ὄντων γέγονεν.

Τρίγωνα δὲ ἰσόπλευρα συνιστάμενα τέτταρα κατὰ σύντρεις ἐπιπέδους γωνίας μίαν στερεὰν γωνίαν ποιεῖ, τῆς ἀμβλυτάτης τῶν ἐπιπέδων γωνιῶν ἐφεξῆς γεγονυῖαν· τοιούτων δὲ ἀποτελεσθεισῶν τεττάρων πρῶτον εἶδος στερεόν, ὅλου περιφεροῦς διανεμητικὸν εἰς ἴσα μέρη καὶ ὅμοια, συνίσταται. δεύτερον δὲ ἐκ μὲν τῶν αὐτῶν τριγώνων, κατὰ δὲ ἰσόπλευρα τρίγωνα ὀκτὼ συστάντων, μίαν ἀπεργασαμένων στερεὰν γωνίαν ἐκ τεττάρων ἐπιπέδων· καὶ γενομένων ἓξ τοιούτων τὸ δεύτερον αὖ σῶμα οὕτως ἔσχε τέλος. τὸ δὲ τρίτον ἐκ δὶς ἑξήκοντα τῶν στοιχείων συμπαγέντων, στερεῶν δὲ γωνιῶν δώδεκα, ὑπὸ πέντε ἐπιπέδων τριγώνων ἰσοπλεύρων περιεχομένης ἑκάστης, εἴκοσι βάσεις ἔχον ἰσοπλεύρους τριγώνους γέγονεν.

Καὶ τὸ μὲν ἕτερον ἀπήλλακτο τῶν στοιχείων

---

[a] As in the accompanying diagram, the triangles AOF, COD, AOE, BOD, COE, BOF are joined together so as to form the equilateral triangle ABC. As Plato has already observed, an equilateral triangle can also be made out of two such triangles.



A. E. Taylor (*A Commentary on Plato's Timaeus*, pp. 374-375), first pointed out the correct meaning of κατὰ διάμετρον, "diagonally." Previously, following Boeckh, editors had supposed that it meant "so that their hypotenuses coincide," *e.g.*, triangle AOF is placed κατὰ διάμετρον with triangle AOE; Plato almost certainly meant that triangles AOF, COD are κατὰ διάμετρον.

tenuses and shorter sides to a common centre, from those triangles, six in number, there is produced one equilateral triangle.[a]

Now when four equilateral triangles are put together so that the three plane angles meet in a point, they make one solid angle, which comes next in order to the most obtuse of the plane angles [b]; and when four such angles are formed, the first solid figure [c] is constructed, dividing the whole of the circumscribed sphere into equal and similar parts. The second solid [d] is formed from the same triangles, but is constructed out of eight equilateral triangles, which make one solid angle from four planes; when six such solid angles have been produced, the second body is in turn completed. The third solid [e] is made up of twice sixty of the elemental triangles and of twelve solid angles, each solid angle being comprised by five plane equilateral triangles, and the manner of its formation gives it twenty equilateral triangular bases.

Now the first of the elemental triangles was dropped

[b] The three plane angles together make two right angles, which is "the most obtuse of the plane angles."

[c] *i.e.*, the regular tetrahedron or pyramid, which has four faces, each an equilateral triangle, and four solid angles, each formed by three of the equilateral triangles: Plato later makes it the element of fire.

[d] *i.e.*, the regular octahedron, which has eight faces, each an equilateral triangle, and six solid angles, each formed by four of the equilateral triangles; Plato later makes it the element of air.

[e] *i.e.*, the icosahedron, which has twenty faces, each an equilateral triangle (and is therefore made up of 120 elemental rectangular scalene triangles, inasmuch as six such triangles are put together to form one equilateral triangle), and twelve solid angles, each formed by five of the equilateral triangles; Plato later made it the element of water.

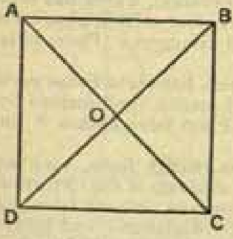
ταῦτα γεννῆσαν, τὸ δὲ ἰσοσκελὲς τρίγωνον ἐγέννα τὴν τοῦ τετάρτου φύσιν, κατὰ τέτταρα συνιστάμενον, εἰς τὸ κέντρον τὰς ὀρθὰς γωνίας συνάγον, ἓν ἰσόπλευρον τετράγωνον ἀπεργασάμενον· ἓξ δὲ τοιαῦτα συμπαγέντα γωνίας ὀκτὼ στερεὰς ἀπετέλεσε, κατὰ τρεῖς ἐπιπέδους ὀρθὰς συναρμοθείσης ἑκάστης· τὸ δὲ σχῆμα τοῦ συστάντος σώματος γέγονε κυβικόν, ἓξ ἐπιπέδους τετραγώνους ἰσοπλεύρους βάσεις ἔχον· ἔτι δὲ οὔσης συστάσεως μιᾶς πέμπτης, ἐπὶ τὸ πᾶν ὁ θεὸς αὐτῇ κατεχρήσατο ἐκεῖνο διαζωγραφῶν.

Iambl. *De Vita Pythag.* 18. 88, ed. Deubner 52. 2-8

Περὶ δ' Ἱππάσου μάλιστα, ὡς ἦν μὲν τῶν Πυθαγορείων, διὰ δὲ τὸ ἐξενεγκεῖν καὶ γράψασθαι πρώτως σφαῖραν τὴν ἐκ τῶν δώδεκα πενταγώνων ἀπώλετο κατὰ θάλατταν ὡς ἀσεβήσας, δόξαν δὲ λάβοι ὡς εὑρών, εἶναι δὲ πάντα ἐκείνου τοῦ ἀνδρός·

---

[a] As in the accompanying figure, the four isosceles scalene triangles AOB, DOC, BOC, DOA placed about the common vertex O form the square ABCD. The fourth figure is the cube, which has six faces, each a square (and is therefore made up of twenty-four of the elemental rectangular isosceles triangles), and eight solid angles, each formed by three of the squares; Plato later makes it the element of earth.



[b] *i.e.*, the regular dodecahedron. This requires, however,

when it had produced these three solids, the nature of the fourth being produced by the isosceles triangle. When four such triangles are joined together, with their right angles drawn towards the centre, they form one equilateral quadrangle[a]; and six such quadrangles, put together, made eight solid angles, each composed of three plane right angles; and the shape of the body thus constructed was cubic, having six plane equilateral quadrangular bases. As there still remained one compound figure, the fifth,[b] God used it for the whole, broidering it with designs.[c]

Iamblichus, *On the Pythagorean Life* 18. 88, ed. Deubner 52. 2-8

It is related of Hippasus that he was a Pythagorean, and that, owing to his being the first to publish and describe the sphere from the twelve pentagons, he perished at sea for his impiety, but he received credit for the discovery, though really it all belonged to

a new element, the regular pentagon. It has twelve faces, each a regular pentagon, and twenty solid angles, each formed by three pentagons. The following passages give evidence that the Pythagoreans may have known the properties of the dodecahedron and pentagon. A number of objects of dodecahedral form have survived from pre-Pythagorean days.

[c] This has often been held, following Plutarch, to refer to the twelve signs of the Zodiac, but A. E. Taylor (*A Commentary on Plato's Timaeus*, p. 377) rightly points out that the dodecagon, not the dodecahedron, would be the appropriate symbol for the Zodiac. He finds a clue to the meaning in *Timaeus Locrus* 98 E, where it is pointed out that of the five regular solids inscribable in the same sphere the dodecahedron has the maximum volume and "comes nearest" to the sphere. Burnet finds the real allusion to the mapping of the apparently spherical heavens into twelve pentagonal regions.

προσαγορεύουσι γὰρ οὕτω τὸν Πυθαγόραν καὶ οὐ καλοῦσιν ὀνόματι.

Luc. *Pro Lapsu inter Salut.* 5, ed. Jacobitz i. 330. 11-14

Καὶ τό γε τριπλοῦν αὐτοῖς τρίγωνον, τὸ δι' ἀλλήλων, τὸ πεντάγραμμον, ᾧ συμβόλῳ πρὸς τοὺς ὁμοδόξους ἐχρῶντο, ὑγίεια πρὸς αὐτῶν ὠνομάζετο.

---

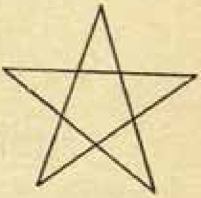
[a] Iamblichus tells the same story, almost word for word, in *De communi Mathematica Scientia* c. 25 (ed. Festa 77. 18-24): the only substantial difference is the substitution of the word ἑξαγώνων for πενταγώνων, which is a slip. The story recalls the passage given above (p. 216) about the Pythagorean who perished at sea for revealing the irrational. He may very well have been the same person as Hippasus, for the irrational would quickly come to light in a study of the regular solids.

HIM (for in this way they refer to Pythagoras, and they do not call him by his name).[a]

Lucian, *On Slips in Greetings* 5, ed. Jacobitz i. 330. 11-14

The triple interlaced triangle, the pentagram, which they (the Pythagoreans) used as a password among members of the same school, was called by them Health.[b]

[b] *Cf.* the scholium to Aristophanes, *Clouds* 609. The pentagram is the star-pentagon, as in the adjoining diagram. The fact that this was a familiar symbol among them lends some plausibility to the belief that they know how to construct the dodecahedron out of twelve pentagons.



VII. DEMOCRITUS

VII. DEMOCRITUS

Plut. *De Comm. Notit.* 39. 3, 1079 E

Ἔτι τοίνυν ὅρα τίνα τρόπον ἀπήντησε Δημοκρίτῳ, διαποροῦντι φυσικῶς καὶ ἐπιτυχῶς, εἰ κῶνος τέμνοιτο παρὰ τὴν βάσιν ἐπιπέδῳ, τί χρὴ διανοεῖσθαι τὰς τῶν τμημάτων ἐπιφανείας, ἴσας ἢ ἀνίσους γινομένας; ἄνισοι μὲν γὰρ οὖσαι τὸν κῶνον ἀνώμαλον παρέξουσι, πολλὰς ἀποχαράξεις λαμβάνοντα βαθμοειδεῖς καὶ τραχύτητας· ἴσων δ' οὐσῶν, ἴσα τμήματα ἔσται, καὶ φανεῖται τὸ τοῦ κυλίνδρου πεπονθὼς ὁ κῶνος, ἐξ ἴσων συγκείμενος καὶ οὐκ ἀνίσων κύκλων, ὅπερ ἐστὶν ἀτοπώτατον.

Archim. *Meth.*, Archim. ed. Heiberg ii. 430. 1-9

Διόπερ καὶ τῶν θεωρημάτων τούτων, ὧν Εὔδοξος ἐξηύρηκεν πρῶτος τὴν ἀπόδειξιν, περὶ τοῦ κώνου καὶ τῆς πυραμίδος, ὅτι τρίτον μέρος ὁ μὲν κῶνος

---

[a] Plutarch tells this on the authority of Chrysippus. Democritus came from Abdera. He was born about the same time as Socrates, and lived to a great age. Plato ignored him in his dialogues, and is said to have wished to burn all his works. The two passages here given contain all that is definitely known of his mathematics, but we are informed that he wrote a book *On the Contact of a Circle and a Sphere*; another on *Geometry*; a third entitled *Geometrica*; a fourth on *Numbers*; a fifth *On Irrational Lines and Solids*; and a sixth called Ἐκπετάσματα, which would deal with the

VII. DEMOCRITUS

Plutarch, *On the Common Notions* 39. 3, 1079 E

CONSIDER further in what manner it occurred to Democritus,[a] in his happy inquiries in natural science, to ask if a cone were cut by a plane parallel to the base,[b] what must we think of the surfaces forming the sections, whether they are equal or unequal? For, if they are unequal, they will make the cone irregular, as having many indentations, like steps, and unevennesses; but if they are equal, the sections will be equal, and the cone will appear to have the property of the cylinder, and to be made up of equal, not unequal, circles, which is very absurd.[c]

Archimedes, *Method*, Archim. ed. Heiberg
ii. 430. 1-9

This is a reason why, in the case of those theorems concerning the cone and pyramid of which Eudoxus first discovered the proof, the theorems that the cone

projection of the armillary sphere on a plane. As his mathematical abilities were obviously great, it is unfortunate that our information is so meagre.

[b] A plane indefinitely near to the base is clearly indicated by what follows.

[c] This bold inquiry first brought the conception of the indefinitely small into Greek mathematics. The story harmonizes with Archimedes' statement that Democritus gave expressions for the volume of the cone and pyramid.

τοῦ κυλίνδρου, ἡ δὲ πυραμὶς τοῦ πρίσματος, τῶν βάσιν ἐχόντων τὴν αὐτὴν καὶ ὕψος ἴσον, οὐ μικρὰν ἀπονείμαι ἄν τις Δημοκρίτῳ μερίδα πρώτῳ τὴν ἀπόφασιν τὴν περὶ τοῦ εἰρημένου σχήματος χωρὶς ἀποδείξεως ἀποφηναμένῳ.

is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, no small share of the credit should be given to Democritus, who was the first to make the assertion with regard to the said figure,[a] though without proof.

[a] So the Greek. Perhaps "type of figure."

VIII. HIPPOCRATES OF CHIOS

VIII. HIPPOCRATES OF CHIOS

(a) General

Philop. *in Phys.* A 2 (Aristot. 185 a 16), ed. Vitelli 31. 3-9

Ἱπποκράτης Χῖός τις ὢν ἔμπορος, λῃστρικῇ νηὶ περιπεσὼν καὶ πάντα ἀπολέσας, ἦλθεν Ἀθήναζε γραψόμενος τοὺς λῃστάς, καὶ πολὺν παραμένων ἐν Ἀθήναις διὰ τὴν γραφὴν χρόνον, ἐφοίτησεν εἰς φιλοσόφους, καὶ εἰς τοσοῦτον ἕξεως γεωμετρικῆς ἦλθεν, ὡς ἐπιχειρῆσαι εὑρεῖν τὸν κύκλου τετραγωνισμόν. καὶ αὐτὸν μὲν οὐχ εὗρε, τετραγωνίσας δὲ τὸν μηνίσκον ᾠήθη ψευδῶς ἐκ τούτου καὶ τὸν κύκλον τετραγωνίζειν· ἐκ γὰρ τοῦ τετραγωνισμοῦ τοῦ μηνίσκου καὶ τὸν τοῦ κύκλου τετραγωνισμὸν ᾠήθη συλλογίζεσθαι.

(b) Quadrature of Lunes

Simpl. *in Phys.* A 2 (Aristot. 185 a 14), ed. Diels 60. 22–68. 32

Ὁ μέντοι Εὔδημος ἐν τῇ Γεωμετρικῇ ἱστορίᾳ οὐκ ἐπὶ τετραγωνικῆς πλευρᾶς δεῖξαί φησι τὸν Ἱπποκράτην τὸν τοῦ μηνίσκου τετραγωνισμόν,

VIII. HIPPOCRATES OF CHIOS

(a) General

Philoponus, *Commentary on Aristotle's Physics* A 2 (185 a 16), ed. Vitelli 31. 3-9

Hippocrates of Chios was a merchant who fell in with a pirate ship and lost all his possessions. He came to Athens to prosecute the pirates and, staying a long time in Athens by reason of the indictment, consorted with philosophers, and reached such proficiency in geometry that he tried to effect the quadrature of the circle. He did not discover this, but having squared the lune he falsely thought from this that he could square the circle also. For he thought that from the quadrature of the lune the quadrature of the circle also could be calculated.[a]

(b) Quadrature of Lunes

Simplicius, *Commentary on Aristotle's Physics* A 2 (185 a 14), ed. Diels 60. 22-68. 32

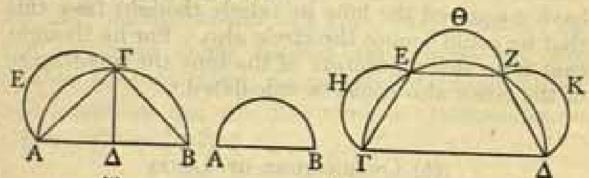
Eudemus, however, in his *History of Geometry* says that Hippocrates did not demonstrate the quadrature

[a] A lune (meniscus) is the figure included between two intersecting arcs of circles. It is unlikely that Hippocrates himself thought he had squared the circle, but for a discussion of this point see *infra*, p. 310 n. *b*.

ἀλλὰ καθόλου, ὡς ἄν τις εἴποι. εἰ γὰρ πᾶς μηνίσκος τὴν ἐκτὸς περιφέρειαν ἢ ἴσην ἔχει ἡμικυκλίου ἢ μείζονα ἢ ἐλάττονα, τετραγωνίζει δὲ ὁ Ἱπποκράτης καὶ τὸν ἴσην ἡμικυκλίου ἔχοντα καὶ τὸν μείζονα καὶ τὸν ἐλάττονα, καθόλου ἂν εἴη δεδειχὼς ὡς δοκεῖ. ἐκθήσομαι δὲ τὰ ὑπὸ τοῦ Εὐδήμου κατὰ λέξιν λεγόμενα ὀλίγα τινὰ προστιθεὶς ⟨εἰς⟩[1] σαφήνειαν ἀπὸ τῆς τῶν Εὐκλείδου Στοιχείων ἀναμνήσεως διὰ τὸν ὑπομνηματικὸν τρόπον τοῦ Εὐδήμου κατὰ τὸ ἀρχαϊκὸν ἔθος συντόμους ἐκθεμένου τὰς ἀποδόσεις. λέγει δὲ ὧδε ἐν τῷ δευτέρῳ βιβλίῳ τῆς Γεωμετρικῆς ἱστορίας.

[1] *εἰς* add. Usener.

[a] As Alexander asserted. Alexander, as quoted by Simplicius *in Phys.* (ed. Diels 56. 1–57. 24), attributes two quadratures to Hippocrates.



(1) (2)

In the first, AB is the diameter of a circle, ΑΓ, ΓΒ are sides of a square inscribed in it, and ΑΕΓ is a semicircle described on ΑΓ. Alexander shows that

lune ΑΕΓ = triangle ΑΓΔ.

In the second, AB is the diameter of semicircle and on ΓΔ, equal to twice AB, a semicircle is described. ΓΕ, ΕΖ, ΖΔ are sides of a regular hexagon, and ΓΗΕ, ΕΘΖ, ΖΚΔ are semicircles described on ΓΕ, ΕΖ, ΖΔ. Alexander shows that

of the lune on the side of a square [a] but generally, as one might say. For every lune has an outer circumference equal to a semicircle or greater or less, and if Hippocrates squared the lune having an outer circumference equal to a semicircle and greater and less, the quadrature would appear to be proved generally. I shall set out what Eudemus wrote word for word, adding only for the sake of clearness a few things taken from Euclid's *Elements* on account of the summary style of Eudemus, who set out his proofs in abridged form in conformity with the ancient practice. He writes thus in the second book of the *History of Geometry*.[b]

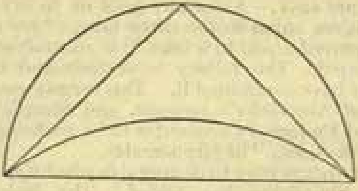
lune ΓΗΕ + lune ΕΘΖ + lune ΖΚΔ + semicircle ΑΒ = trapezium ΓΕΖΔ.

The proofs are easy. Alexander goes on to say that if the rectilineal figure equal to the three lunes ("for a rectilineal figure was proved equal to a lune") is subtracted, the circle will be squared. The fallacy is obvious and Hippocrates could hardly have committed it. This throws some doubt on the whole of Alexander's account, and Simplicius himself observes that Eudemus's account is to be preferred as he was "nearer to the times" of Hippocrates.

[b] It is not always easy to distinguish what Eudemus wrote and what Simplicius has added. The task was first attempted by Allman (*Hermathena* iv., pp. 180-228; *Greek Geometry from Thales to Euclid*, pp. 64-75). Diels, in his edition of Simplicius published in 1882, with the help of Usener, printed in spaced type what they attributed to Eudemus. In 1883 Tannery (*Mémoires scientifiques* i., pp. 339-370) edited what he thought the Eudemian passages. Heiberg (*Philologus* xliii., pp. 336-344) gave his views in 1884. Rudio discussed the question exhaustively in 1907 (*Der Bericht des Simplicius über die Quadraturen des Antiphon und Hippokrates*), but unfortunately his judgement is not always trustworthy. Heath (*H.G.M.* i. 183-200) has an excellent analysis. In the following pages I have given only such passages as can safely be attributed to Eudemus and omitted the rest.

"Καὶ οἱ τῶν μηνίσκων δὲ τετραγωνισμοὶ δόξαντες εἶναι τῶν οὐκ ἐπιπολαίων διαγραμμάτων διὰ τὴν οἰκειότητα τὴν πρὸς τὸν κύκλον ὑφ' Ἱπποκράτους ἐγράφησάν τε πρώτου καὶ κατὰ τρόπον ἔδοξαν ἀποδοθῆναι· διόπερ ἐπὶ πλέον ἁψώμεθά τε καὶ διέλθωμεν. ἀρχὴν μὲν οὖν ἐποιήσατο καὶ πρῶτον ἔθετο τῶν πρὸς αὐτοὺς χρησίμων, ὅτι τὸν αὐτὸν λόγον ἔχει τά τε ὅμοια τῶν κύκλων τμήματα πρὸς ἄλληλα καὶ αἱ βάσεις αὐτῶν δυνάμει. τοῦτο δὲ ἐδείκνυεν ἐκ τοῦ τὰς διαμέτρους δεῖξαι τὸν αὐτὸν λόγον ἐχούσας δυνάμει τοῖς κύκλοις.

"Δειχθέντος δὲ αὐτῷ τούτου πρῶτον μὲν ἔγραφε μηνίσκου τὴν ἐκτὸς περιφέρειαν ἔχοντος ἡμικυκλίου



τίνα τρόπον γένοιτο ἂν τετραγωνισμός. ἀπεδίδου δὲ τοῦτο περὶ τρίγωνον ὀρθογώνιόν τε καὶ ἰσοσκελὲς ἡμικύκλιον περιγράψας καὶ περὶ τὴν βάσιν τμῆμα κύκλου τοῖς ὑπὸ τῶν ἐπιζευχθεισῶν ἀφαιρουμένοις ὅμοιον. ὄντος δὲ τοῦ περὶ τὴν βάσιν τμήματος ἴσου τοῖς περὶ τὰς ἑτέρας ἀμφοτέροις, καὶ κοινοῦ προστεθέντος τοῦ μέρους τοῦ τριγώνου τοῦ ὑπὲρ τὸ τμῆμα τὸ περὶ τὴν βάσιν, ἴσος ἔσται ὁ μηνίσκος τῷ τριγώνῳ. ἴσος οὖν ὁ μηνίσκος τῷ τριγώνῳ δειχθεὶς τετραγωνίζοιτο ἄν. οὕτως μὲν

"The quadratures of lunes, which seemed to belong to an uncommon class of propositions by reason of the close relationship to the circle, were first investigated by Hippocrates, and seemed to be set out in correct form; therefore we shall deal with them at length and go through them. He made his starting-point, and set out as the first of the theorems useful to his purpose, that similar segments of circles have the same ratios as the squares on their bases.[a] And this he proved by showing that the squares on the diameters have the same ratios as the circles.[b]

"Having first shown this he described in what way it was possible to square a lune whose outer circumference was a semicircle. He did this by circumscribing about a right-angled isosceles triangle a semicircle and about the base a segment of a circle similar to those cut off by the sides.[c] Since the segment about the base is equal to the sum of those about the sides, it follows that when the part of the triangle above the segment about the base is added to both the lune will be equal to the triangle. Therefore the lune, having been proved equal to the triangle, can be squared. In this way, taking

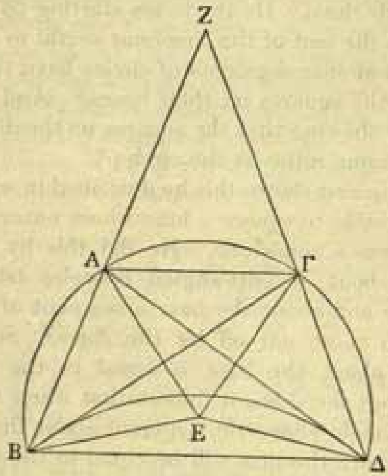
[a] Lit. "as the bases in square."

[b] This is Eucl. xii. 2 (see *infra*, pp. 458-465). Euclid proves it by a method of exhaustion, based on a lemma or its equivalent which, on the evidence of Archimedes himself, can safely be attributed to Eudoxus. We are not told how Hippocrates effected the proof.

[c] As Simplicius notes, this is the problem of Eucl. iii. 33 and involves the knowledge that similar segments contain equal angles.

οὖν ἡμικυκλίου τὴν ἔξω τοῦ μηνίσκου περιφέρειαν ὑποθέμενος ἐτετραγώνισεν ὁ Ἱπποκράτης τὸν μηνίσκον εὐκόλως.

"Εἶτα ἐφεξῆς μείζονα ἡμικυκλίου ὑποτίθεται συστησάμενος τραπέζιον τὰς μὲν τρεῖς ἔχον πλευρὰς



ἴσας ἀλλήλαις, τὴν δὲ μίαν τὴν μείζω τῶν παραλλήλων τριπλασίαν ἐκείνων ἑκάστης δυνάμει, καὶ τό τε τραπέζιον περιλαβὼν κύκλῳ καὶ περὶ τὴν μεγίστην αὐτοῦ πλευρὰν ὅμοιον τμῆμα περιγράψας τοῖς ὑπὸ τῶν ἴσων τριῶν ἀποτεμνομένοις ἀπὸ τοῦ κύκλου. ὅτι δὲ μεῖζόν ἐστιν ἡμικυκλίου τὸ λεχθὲν τμῆμα, δῆλον ἀχθείσης ἐν τῷ τραπεζίῳ διαμέτρου. ἀνάγκη γὰρ ταύτην ὑπὸ δύο πλευρὰς ὑποτείνουσαν τοῦ τραπεζίου τῆς ὑπολοίπου μιᾶς μείζονα ἢ δι-

a semicircle as the outer circumference of the lune, Hippocrates readily squared the lune.

"Next in order he assumes [an outer circumference] greater than a semicircle [obtained by] constructing a trapezium having three sides equal to one another while one, the greater of the parallel sides, is such that the square on it is three times the square on each of those sides, and then comprehending the trapezium in a circle and circumscribing about [a] its greatest side a segment similar to those cut off from the circle by the three equal sides.[b] That the said segment [c] is greater than a semicircle is clear if a diagonal is drawn in the trapezium. For this diagonal, subtending two sides of the trapezium, must be such that the square on it is greater than double the square on

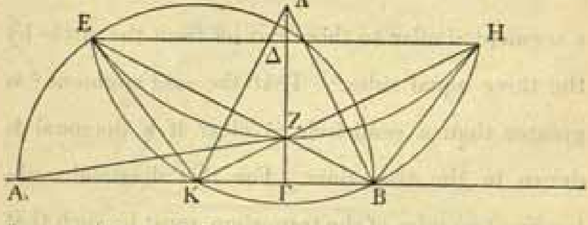
[a] *i.e.* "describing on."

[b] Simplicius here inserts a proof that a circle can be described about the trapezium.

[c] *i.e.*, the segment bounded by the outer circumference. Eudemus is going to show that the angle in it is acute and therefore the segment is greater than a semicircle.

πλασίαν εἶναι δυνάμει. ἡ ἄρα ΒΓ μεῖζον ἢ διπλάσιον δύναται ἑκατέρας τῶν ΒΑ, ΑΓ, ὥστε καὶ τῆς ΓΔ. καὶ τὴν μεγίστην ἄρα τῶν τοῦ τραπεζίου πλευρῶν τὴν ΒΔ ἀναγκαῖον ἔλαττον δύνασθαι τῆς τε διαμέτρου καὶ τῶν ἑτέρων πλευρῶν ἐκείνης, ὑφ' ἣν ὑποτείνει μετὰ τῆς διαμέτρου ἡ λεχθεῖσα. αἱ γὰρ ΒΓ, ΓΔ μεῖζον ἢ τριπλάσιον δύνανται τῆς ΓΔ, ἡ δὲ ΒΔ τριπλάσιον. ὀξεῖα ἄρα ἐστὶν ἡ ἐπὶ τῆς μείζονος τοῦ τραπεζίου πλευρᾶς βεβηκυῖα γωνία. μεῖζον ἄρα ἡμικυκλίου ἐστὶ τὸ τμῆμα ἐν ᾧ ἐστιν. ὅπερ ἐστὶν ἡ ἔξω περιφέρεια τοῦ μηνίσκου.

"Εἰ δὲ ἐλάττων ἡμικυκλίου εἴη, προγράψας τοιόνδε τι ὁ Ἱπποκράτης τοῦτο κατεσκεύασεν·



ἔστω κύκλος οὗ διάμετρος ἐφ' ᾗ [ἡ][1] ΑΒ, κέντρον δὲ αὐτοῦ ἐφ' ᾧ Κ· καὶ ἡ μὲν ἐφ' ᾗ ΓΔ δίχα τε καὶ πρὸς ὀρθὰς τεμνέτω τὴν ἐφ' ᾗ ΒΚ· ἡ δὲ ἐφ' ᾗ ΕΖ κείσθω ταύτης μεταξὺ καὶ τῆς περιφερείας ἐπὶ τὸ Β νεύουσα τῶν ἐκ τοῦ κέντρου ἡμιολία οὖσα

[1] ἡ om. Diels.

[a] A proof is supplied in the text, probably by Simplicius though Diels attributes it to Eudemus. The proof is that, since ΒΔ is parallel to ΑΓ but greater than it, ΔΓ and ΒΑ produced will meet in Ζ. Then ΖΑΓ is an isosceles triangle,

one of the remaining sides. Therefore the square on BΓ is greater than double the square on either BA, AΓ, and therefore also on ΓΔ.[a] Therefore the square on BΔ, the greatest of the sides of the trapezium, must be less than the sum of the squares on the diagonal and that one of the other sides which is subtended by the said [greatest] side together with the diagonal.[b] For the squares on BΓ, ΓΔ are greater than three times, and the square on BΔ is equal to three times, the square on ΓΔ. Therefore the angle standing on the greatest side of the trapezium[c] is acute. Therefore the segment in which it is is greater than a semicircle. And this segment is the outer circumference of the lune.[d]

" If [the outer circumference] were less than a semicircle, Hippocrates solved[e] this also, using the following preliminary construction. Let there be a circle with diameter AB and centre K. Let ΓΔ bisect BK at right angles; and let the straight line EZ be placed between this and the circumference verging towards B so that the square on it is one-and-a-half

so that the angle ZAΓ is acute, and therefore the angle BAΓ is obtuse.

[b] *i.e.* $B\Delta^2 < B\Gamma^2 + \Gamma\Delta^2$.

[c] *i.e.* the angle BΓΔ.

[d] Simplicius notes that Eudemus has omitted the actual squaring of the lune, presumably as being obvious. Since

$$B\Delta^2 = 3BA^2$$

(segment on BΔ) = 3 (segment on BA)

= sum of segments on BA, AΓ, ΓΔ.

Adding to each side of the equation the portion of the trapezium included by the sides BA, AΓ and ΓΔ and the circumference of the segment on BΔ, we get

trapezium ABΔΓ = lune bounded by the two circumferences

and so the lune is " squared."

[e] Lit. " constructed."

δυνάμει. ἡ δὲ ἐφ' ἧ ΕΗ ἤχθω παρὰ τὴν ἐφ' ἧ ΑΒ, καὶ ἀπὸ τοῦ Κ ἐπεζεύχθωσαν ἐπὶ τὰ Ε, Ζ. συμπιπτέτω δὲ ἐκβαλλομένη ἡ ἐπὶ τὸ Ζ ἐπιζευχθεῖσα τῇ ἐφ' ἧ ΕΗ κατὰ τὸ Η καὶ πάλιν ἀπὸ τοῦ Β ἐπὶ τὰ Ζ, Η ἐπεζεύχθωσαν. φανερὸν δὴ ὅτι ἡ μὲν ἐφ' ἧ ΕΖ ἐκβαλλομένη ἐπὶ τὸ Β πεσεῖται (ὑπόκειται γὰρ ἡ ΕΖ ἐπὶ τὸ Β νεύουσα), ἡ δὲ ἐφ' ἧ ΒΗ ἴση ἔσται τῇ ἐφ' ἧ ΕΚ.

"Τούτων οὖν οὕτως ἐχόντων τὸ τραπέζιόν φημι ἐφ' οὗ ΕΚΒΗ περιλήψεται κύκλος.

"Περιγεγράφθω[1] δὴ περὶ τὸ ΕΖΗ τρίγωνον τμῆμα κύκλου, δῆλον ὅτι ἑκάτερον τῶν ΕΖ, ΖΗ ὅμοιον ἑκάστῳ τῶν ΕΚ, ΚΒ, ΒΗ τμημάτων.

"Τούτων οὕτως ἐχόντων ὁ γενόμενος μηνίσκος οὗ ἐκτὸς περιφέρεια ἡ ΕΚΒΗ ἴσος ἔσται τῷ εὐθυγράμμῳ τῷ συγκειμένῳ ἐκ τῶν τριῶν τριγώνων τῶν ΒΖΗ, ΒΖΚ, ΕΚΖ. τὰ γὰρ ἀπὸ τῶν εὐθειῶν ἐφ' αἷς ΕΖ, ΖΗ ἀφαιρούμενα ἐντὸς τοῦ μηνίσκου ἀπὸ τοῦ εὐθυγράμμου τμήματα ἴσα ἐστὶ τοῖς ἐκτὸς

[1] Περιγεγράφθω . . . τμημάτων. In the text of Simplicius this sentence precedes the one above and Simplicius's comments thereon. It is here restored to the place which it must have occupied in Eudemus's *History*.

---

[a] This is the first example we have had to record of the type of construction known to the Greeks as νεύσεις, *inclinations* or *vergings*. The general problem is to place a straight line so as to *verge towards* (pass through) a given point and so that a given length is intercepted on it by other lines. In this case the problem amounts to finding a length x such that, if Z be taken on ΓΔ so that $BZ = x$ and BZ be produced to

times the square on one of the radii.[a] Let EH be drawn parallel to AB, and from K let [straight lines] be drawn joining E and Z. Let the straight line [KZ] joined to Z and produced meet EH at H, and again let [straight lines] be drawn from B joining Z and H. It is then manifest that EZ produced will pass through B—for by hypothesis EZ verges towards B—and BH will be equal to EK.

"This being so, I say that the trapezium EKBH can be comprehended in a circle.

"Next let a segment of a circle be circumscribed about the triangle EZH; then clearly each of the segments on EZ, ZH will be similar to the segments on EK, KB, BH.

"This being so, the lune so formed, whose outer circumference is EKBH, will be equal to the rectilineal figure composed of the three triangles BZH, BZK, EKZ. For the segments cut off from the rectilineal figure, inside the lune, by the straight lines EZ, ZH are (together) equal to the segments outside

meet the circumference in E, then $EZ^2 = \frac{3}{2}AK^2$, or $EZ = \sqrt{\frac{3}{2}}$ AK. If this is done, $EB \,.\, BZ = AB \,.\, B\Gamma = AK^2$

or $$(x + \sqrt{\tfrac{3}{2}}\,a) \,.\, x = a^2, \text{ where } AK = a.$$

In other words, the problem amounts to solving the quadratic equation $$x^2 + \sqrt{\tfrac{3}{2}}\,ax = a^2.$$

This would be recognized by the Greeks as the problem of "applying to a straight line of length $\sqrt{\frac{3}{2}} \,.\, a$, a rectangle exceeding by a square figure and equal in area to a^2," and could have been solved theoretically by the Pythagorean method preserved in Eucl. ii. 6. Was this the method used by Hippocrates? Though it may have been, the authorities prefer to believe he used mechanical means (*H.G.M.* i. 196, Rudio, *loc cit.*, p. 59, Zeuthen, *Geschichte d. Math.*, p. 80). He could have marked on a ruler a length equal to $\sqrt{\frac{3}{2}}$ AK and moved it about until it was in the required position.

τοῦ εὐθυγράμμου τμήμασιν ἀφαιρουμένοις ὑπὸ τῶν ΕΚ, ΚΒ, ΒΗ. ἑκάτερον γὰρ τῶν ἐντὸς ἡμιόλιόν ἐστιν ἑκάστου τῶν ἐκτός. ἡμιολία γὰρ[1] ὑπόκειται ἡ ΕΖ τῆς ἐκ τοῦ κέντρου, τουτέστι τῆς ΕΚ καὶ ΚΒ καὶ ΒΗ. εἰ οὖν ὁ μὲν μηνίσκος τὰ τρία τμήματά ἐστι καὶ τοῦ εὐθυγράμμου τὸ παρὰ τὰ δύο τμήματα, τὸ δὲ εὐθύγραμμον μετὰ τῶν δύο τμημάτων ἐστὶ χωρὶς τῶν τριῶν, ἔστι δὲ τὰ δύο τμήματα τοῖς τρισὶν ἴσα, ἴσος ἂν εἴη ὁ μηνίσκος τῷ εὐθυγράμμῳ.

"Ὅτι δὲ οὗτος ὁ μηνίσκος ἐλάττονα ἡμικυκλίου τὴν ἐκτὸς ἔχει περιφέρειαν, δείκνυσι διὰ τοῦ τὴν ΕΚΗ γωνίαν ἐν τῷ ἐκτὸς οὖσαν τμήματι ἀμβλεῖαν εἶναι. ὅτι δε ἀμβλεῖά ἐστιν ἡ ὑπὸ ΕΚΗ γωνία, δείκνυσιν οὕτως· ἐπεὶ[2] ἡ μὲν ἐφ' ᾗ ΕΖ ἡμιολία ἐστὶ τῶν ἐκ τοῦ κέντρου δυνάμει, ἡ δὲ ἐφ' ᾗ ΚΒ μείζων τῆς ἐφ' ᾗ ΒΖ ἢ διπλασία δυνάμει, φανερὸν ὅτι καὶ ἡ ἐφ' ᾗ ΚΕ ἔσται τῆς ἐφ' ᾗ ΚΖ ἄρα μείζων ἢ διπλασία δυνάμει. ἡ δὲ ἐφ' ᾗ ΕΖ μείζων ἐστὶ δυνάμει τῶν ἐφ' αἷς ΕΚ, ΚΖ. ἀμβλεῖα ἄρα ἐστὶν ἡ πρὸς τῷ Κ γωνία, ἔλαττον ἄρα ἡμικυκλίου τὸ τμῆμα ἐν ᾧ ἐστιν.

"Οὕτως μὲν οὖν ὁ Ἱπποκράτης πάντα μηνίσκον ἐτετραγώνισεν, εἴπερ καὶ τὸν ἡμικυκλίου καὶ τὸν

[1] δυνάμει must be understood after ἡμιολία γάρ, as Bretschneider first pointed out, but Diels and Rudio think that Simplicius probably omitted it as obvious, here and in his own comments.

[2] ἐπεὶ . . . ἐστιν. Eudemus purports to give the proof in Hippocrates' own words. Unfortunately Simplicius's version is too confused to be worth reproducing. The proof is here given as reconstructed by Rudio. That it is substantially the proof given by Hippocrates is clear.

the rectilineal figure cut off by EK, KB, BH. For each of the inner segments is one-and-a-half times each of the outer, because, by hypothesis, the square on EZ is one-and-a-half times the square on the radius, that is, the square on EK or KB or BH. Inasmuch then as the lune is made up of the three segments and the rectilineal figure *less* the two segments—the rectilineal figure including the two segments but not the three—while the sum of the two segments is equal to the sum of the three, it follows that the lune is equal to the rectilineal figure.

"That this lune has its outer circumference less than a semicircle, he proves by means of the angle EKH in the outer segment being obtuse. And that the angle EKH is obtuse, he proves thus.

Since $\mathrm{EZ}^2 = \frac{3}{2}\,\mathrm{EK}^2$

and [a] $\mathrm{KB}^2 > 2\mathrm{BZ}^2$,

it is manifest that $\mathrm{EK}^2 > 2\mathrm{KZ}^2$.

Therefore $\mathrm{EZ}^2 > \mathrm{EK}^2 + \mathrm{KZ}^2$.

The angle at K is therefore obtuse, so that the segment in which it is is less than a semicircle.

"Thus Hippocrates squared every lune, seeing that [he squared] not only the lune which has for its outer circumference a semicircle, but also the lune in which

[a] This is assumed. Heath (*H.G.M.* i. 195) supplies the following proof:

By hypothesis, $\mathrm{EZ}^2 = \frac{3}{2}\mathrm{KB}^2$.

Also, since A, E, Z, Γ are concyclic,

$$\mathrm{EB} \,.\, \mathrm{BZ} = \mathrm{AB} \,.\, \mathrm{B\Gamma} = \mathrm{KB}^2$$

or $$\mathrm{EZ} \,.\, \mathrm{ZB} + \mathrm{BZ}^2 = \mathrm{KB}^2 = \frac{2}{3}\mathrm{EZ}^2.$$

It follows that $\mathrm{EZ} > \mathrm{ZB}$ and that $\mathrm{KB}^2 > 2\mathrm{BZ}^2$.

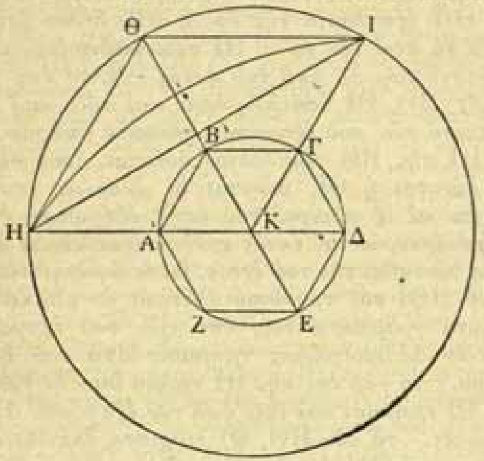
μείζονα ἡμικυκλίου καὶ τὸν ἐλάττονα ἔχοντα τὴν ἐκτὸς περιφέρειαν.

" Ἀλλὰ μηνίσκον ἅμα καὶ κύκλον ἐτετραγώνισεν οὕτως· ἔστωσαν περὶ κέντρον ἐφ' οὗ Κ δύο κύκλοι, ἡ δὲ τοῦ ἐκτὸς διάμετρος ἑξαπλασία δυνάμει τῆς τοῦ ἐντὸς καὶ ἑξαγώνου ἐγγραφέντος εἰς τὸν ἐντὸς κύκλον τοῦ ἐφ' οὗ ΑΒΓΔΕΖ αἵ τε ἐφ' ὧν ΚΑ, ΚΒ, ΚΓ ἐκ τοῦ κέντρου ἐπιζευχθεῖσαι ἐκβεβλήσθωσαν ἕως τῆς τοῦ ἐκτὸς κύκλου περιφερείας καὶ αἱ ἐφ' ὧν ΗΘ, ΘΙ, ⟨ΗΙ⟩[1] ἐπεζεύχθωσαν καὶ δῆλον ὅτι καὶ αἱ ΗΘ, ΘΙ ἑξαγώνου εἰσὶ πλευραὶ τοῦ εἰς τὸν μείζονα κύκλον ἐγγραφομένου. καὶ περὶ τὴν ἐφ' ᾗ ΗΙ τμῆμα ὅμοιον τῷ ἀφαιρουμένῳ ὑπὸ τῆς ἐφ' ᾗ ΗΘ περιγεγράφθω. ἐπεὶ οὖν τὴν μὲν ἐφ' ᾗ ΗΙ τριπλασίαν ἀνάγκη εἶναι δυνάμει τῆς ἐφ' ᾗ ΘΗ τοῦ ἑξαγώνου πλευρᾶς (ἡ γὰρ ὑπὸ δύο τοῦ ἑξαγώνου πλευρὰς ὑποτείνουσα μετὰ ἄλλης μιᾶς ὀρθὴν περι-

[1] ΗΙ add. Usener.

the outer circumference is greater, and that in which it is less, than a semicircle.

" But he also squared a lune and a circle together in the following manner. Let there be two circles



with K as centre, such that the square on the diameter of the outer is six times the square on the diameter of the inner. Let a [regular] hexagon ΑΒΓΔΕΖ be inscribed in the inner circle, and let KA, KB, KΓ be joined from the centre and produced as far as the circumference of the outer circle, and let HΘ, ΘI, HI be joined. Then it is clear that HΘ, ΘI are sides of a [regular] hexagon inscribed in the outer circle. About HI let a segment be circumscribed similar to the segment cut off by HΘ. Since then $HI^2 = 3\Theta H^2$ (for the square on the line subtended by two sides of the hexagon, together with the square on one other

ἔχουσα γωνίαν τὴν ἐν ἡμικυκλίῳ ἴσον δύναται τῇ διαμέτρῳ, ἡ δὲ διάμετρος τετραπλάσιον δύναται τῆς τοῦ ἑξαγώνου ἴσης οὔσης τῇ ἐκ τοῦ κέντρου διὰ τὸ τὰ μήκει διπλάσια εἶναι δυνάμει τετραπλάσια), ἡ δὲ ΘΗ ἑξαπλασία τῆς ἐφ' ᾗ ΑΒ, δῆλον ὅτι τὸ τμῆμα τὸ περὶ τὴν ἐφ' ᾗ ΗΙ περιγραφὲν ἴσον εἶναι συμβαίνει τοῖς τε ἀπὸ τοῦ ἐκτὸς κύκλου ὑπὸ τῶν ἐφ' αἷς ΗΘ, ΘΙ ἀφαιρουμένοις καὶ τοῖς ἀπὸ τοῦ ἐντὸς ὑπὸ τῶν τοῦ ἑξαγώνου πλευρῶν ἁπασῶν. ἡ γὰρ ΗΙ τῆς ΗΘ τριπλάσιον δύναται, ἴσον δὲ τῇ ΗΘ δύναται ἡ ΘΙ, δύναται δὲ ἑκατέρα τούτων ἴσον καὶ αἱ ἓξ πλευραὶ τοῦ ἐντὸς ἑξαγώνου, διότι καὶ ἡ διάμετρος τοῦ ἐκτὸς κύκλου ἑξαπλάσιον ὑπόκειται δύνασθαι τῆς τοῦ ἐντός, ὥστε ὁ μὲν μηνίσκος ἐφ' οὗ ΗΘΙ τοῦ τριγώνου ἐλάττων ἂν εἴη ἐφ' οὗ τὰ αὐτὰ γράμματα τοῖς ὑπὸ τῶν τοῦ ἑξαγώνου πλευρῶν ἀφαιρουμένοις τμήμασιν ἀπὸ τοῦ ἐντὸς κύκλου. τὸ γὰρ ἐπὶ τῆς ΗΙ τμῆμα ἴσον ἦν τοῖς τε ΗΘ, ΘΙ τμήμασι καὶ τοῖς ὑπὸ τοῦ ἑξαγώνου ἀφαιρουμένοις. τὰ οὖν ΗΘ, ΘΙ τμήματα ἐλάττω ἐστὶ τοῦ περὶ τὴν ΗΙ ⟨τμήματος τοῖς⟩[1] τμήμασι [καὶ][2] τοῖς ὑπὸ τοῦ ἑξαγώνου ἀφαιρουμένοις. κοινοῦ οὖν προστεθέντος τοῦ ὑπὲρ τὸ τμῆμα τὸ περὶ τὴν ΗΙ μέρους τοῦ τριγώνου, ἐκ μὲν τούτου καὶ τοῦ περὶ τὴν ΗΙ τμήματος τὸ τρίγωνον ἔσται, ἐκ δὲ τοῦ αὐτοῦ καὶ τῶν ΗΘ, ΘΙ τμημάτων ὁ μηνίσκος. ἔσται οὖν ἐλάττων ὁ μηνίσκος τοῦ τριγώνου τοῖς ὑπὸ τοῦ ἑξαγώνου ἀφαιρουμένοις τμήμασιν. ὁ ἄρα

[1] τμήματος τοῖς add. Bretschneider.
[2] καὶ om. Bretschneider.

[a] If ΗΛ be a side of the hexagon, then ΙΛ is a diameter and the angle ΙΗΛ is right. Therefore $ΗΙ^2+ΗΛ^2=ΙΛ^2$,

side, is equal, since they form a right angle in the semicircle, to the square on the diameter, and the square on the diameter is four times the side of the hexagon, the diameter being twice the side in length and so four times as great in square [a]), and $\Theta H^2 = 6\ AB^2$, it is manifest that the segment circumscribed about HI is equal to the segments cut off from the outer circle by HΘ, ΘI, together with the segments cut off from the inner circle by all the sides of the hexagon.[b] For $HI^2 = 3\ H\Theta^2$, and $\Theta I^2 = H\Theta^2$, while ΘI^2 and $H\Theta^2$ are each equal to the sum of the squares on the six sides of the inner hexagonal, since, by hypothesis, the diameter of the outer circle is six times that of the inner. Therefore the lune HΘI is smaller than the triangle HΘI by the segments taken away from the inner circle by the sides of the hexagon. For the segment on HI is equal to the sum of the segments on HΘ, ΘI and those taken away by the hexagon. Therefore the segments [on] HΘ, ΘI are less than the segment about HI by the segments taken away by the hexagon. If to both sides there is added the part of the triangle which is above the segment about HI,[c] out of this and the segment about HI will be formed the triangle, while out of the latter and the segments [on] HΘ, ΘI will be formed the lune. Therefore the lune will be less than the triangle by the segments taken away by the hexagon. For the lune and the

and so $HI^2 + \Theta H^2 = I\Lambda^2 = 4\Theta H^2$ (since $I\Lambda = 2\Theta H$). Consequently $HI^2 = 3\Theta H^2$.

[b] For (segment on HI) = 3 (segment on HΘ)
= 2 (segment on HΘ) + 6 (segment on AB)
= (segments on HΘ, ΘI) + (all segments of inner circle).

[c] *i.e.*, the figure bounded by HΘ, ΘI and the arc IH.

μηνίσκος καὶ τὰ ὑπὸ τοῦ ἑξαγώνου ἀφαιρούμενα τμήματα ἴσα ἐστὶν τῷ τριγώνῳ. καὶ κοινοῦ προστεθέντος τοῦ ἑξαγώνου τὸ τρίγωνον τοῦτο καὶ τὸ ἑξάγωνον ἴσα ἐστὶ τῷ τε μηνίσκῳ τῷ λεχθέντι καὶ τῷ κύκλῳ τῷ ἐντός. εἰ οὖν τὰ εἰρημένα εὐθύγραμμα δυνατὸν τετραγωνισθῆναι, καὶ τὸν κύκλον ἄρα μετὰ τοῦ μηνίσκου."

(c) Two Mean Proportionals

Procl. *in Eucl.* i., ed. Friedlein 212. 24-213. 11

Ἡ δὲ ἀπαγωγὴ μετάβασίς ἐστιν ἀπ' ἄλλου προβλήματος ἢ θεωρήματος ἐπ' ἄλλο, οὗ γνωσθέντος ἢ πορισθέντος καὶ τὸ προκείμενον ἔσται καταφανές, οἷον ὥσπερ καὶ τοῦ διπλασιασμοῦ τοῦ κύβου ζητηθέντος μετέθεσαν τὴν ζήτησιν εἰς ἄλλο, ᾧ τοῦτο ἕπεται, τὴν εὕρεσιν τῶν δύο μέσων, καὶ τὸ λοιπὸν ἐζήτουν, πῶς ἂν δύο δοθεισῶν εὐθειῶν δύο μέσαι ἀνάλογον εὑρεθεῖεν. πρῶτον δέ φασι τῶν ἀπορουμένων διαγραμμάτων τὴν ἀπαγωγὴν ποιήσασθαι Ἱπποκράτην τὸν Χῖον, ὃς καὶ μηνίσκον ἐτετραγώνισε καὶ ἄλλα πολλὰ κατὰ γεωμετρίαν εὗρεν εὐφυὴς περὶ τὰ διαγράμματα εἴπερ τις ἄλλος γενόμενος.

---

[a] What Hippocrates showed was that if $\frac{a}{x}=\frac{x}{y}=\frac{y}{b}$, then

segments taken away by the hexagon are equal to the triangle. When the hexagon is added to both sides, this triangle and the hexagon will be equal to the aforesaid lune and to the inner circle. If then the aforementioned rectilineal figures can be squared, so also can the circle with the lune."

(c) Two Mean Proportionals

Proclus, *on Euclid* i., ed. Friedlein 212. 24-213. 11

Reduction is a transition from one problem or theorem to another, whose solution or construction makes manifest also that which is propounded, as when those who sought to double the cube transferred the investigation to another [problem] which it follows, the discovery of the two means, and from that time forward inquired how between two given straight lines two mean proportionals could be found. They say the first to effect the reduction of the difficult constructions was Hippocrates of Chios, who also squared a lune and discovered many other things in geometry, being unrivalled in the cleverness of his constructions.[a]

$\frac{a^3}{x^3}=\frac{a}{b}$, so that if $b=2a$, a cube of side x is twice the size of a cube of side a. For a fuller discussion, see *infra*, p. 258 n. *b*. It has been supposed from this passage that Hippocrates discovered the method of geometrical reduction, but this is unlikely.

IX. SPECIAL PROBLEMS

IX. SPECIAL PROBLEMS

1. DUPLICATION OF THE CUBE

(a) General

Theon Smyr., ed. Hiller 2. 3-12

Ἐρατοσθένης μὲν γὰρ ἐν τῷ ἐπιγραφομένῳ Πλατωνικῷ φησιν ὅτι, Δηλίοις τοῦ θεοῦ χρήσαντος ἐπὶ ἀπαλλαγῇ λοιμοῦ βωμὸν τοῦ ὄντος διπλασίονα κατασκευάσαι, πολλὴν ἀρχιτέκτοσιν ἐμπεσεῖν ἀπορίαν ζητοῦσιν ὅπως χρὴ στερεὸν στερεοῦ γενέσθαι διπλάσιον, ἀφικέσθαι τε πευσομένους περὶ τούτου Πλάτωνος. τὸν δὲ φάναι αὐτοῖς, ὡς ἄρα οὐ διπλασίου βωμοῦ ὁ θεὸς δεόμενος τοῦτο Δηλίοις ἐμαντεύσατο, προφέρων δὲ καὶ ὀνειδίζων τοῖς Ἕλλησιν ἀμελοῦσι μαθημάτων καὶ γεωμετρίας ὠλιγωρηκόσιν.

Eutoc. *Comm. in Archim. de Sphaera et Cyl.* ii., Archim. ed. Heiberg iii. 88. 4–90. 13

Βασιλεῖ Πτολεμαίῳ Ἐρατοσθένης χαίρειν.

Τῶν ἀρχαίων τινὰ τραγῳδοποιῶν φασιν εἰσαγαγεῖν τὸν Μίνω τῷ Γλαύκῳ κατασκευάζοντα τάφον,

[a] Wilamowitz (*Gött. Nachr.*, 1894) shows that the letter is a forgery, but there is no reason to doubt the story it relates, which is indeed amply confirmed; and the author must be thanked for having included in his letter a proof and an

IX. SPECIAL PROBLEMS

1. DUPLICATION OF THE CUBE

(a) General

Theon of Smyrna, ed. Hiller 2. 3-12

In his work entitled *Platonicus* Eratosthenes says that, when the god announced to the Delians by oracle that to get rid of a plague they must construct an altar double of the existing one, their craftsmen fell into great perplexity in trying to find how a solid could be made double of another solid, and they went to ask Plato about it. He told them that the god had given this oracle, not because he wanted an altar of double the size, but because he wished, in setting this task before them, to reproach the Greeks for their neglect of mathematics and their contempt for geometry.

Eutocius, *Commentary on Archimedes' Sphere and Cylinder* ii., Archim. ed. Heiberg iii. 88. 4–90. 13

To King Ptolemy Eratosthenes sends greeting.[a]

They say that one of the ancient tragic poets represented Minos as preparing a tomb for Glaucus,

epigram, taken from a votive monument, which are the genuine work of Eratosthenes (*infra*, pp. 294-297). The monarch addressed is Ptolemy Euergetes, to whose son, Philopator, Eratosthenes was tutor.

πυθόμενον δέ, ὅτι πανταχοῦ ἑκατόμπεδος εἴη, εἰπεῖν·

μικρόν γ' ἔλεξας βασιλικοῦ σηκὸν τάφου·
διπλάσιος ἔστω, τοῦ καλοῦ δὲ μὴ σφαλεὶς
δίπλαζ' ἕκαστον κῶλον ἐν τάχει τάφου.

ἐδόκει δὲ διημαρτηκέναι· τῶν γὰρ πλευρῶν διπλασιασθεισῶν τὸ μὲν ἐπίπεδον γίνεται τετραπλάσιον, τὸ δὲ στερεὸν ὀκταπλάσιον. ἐζητεῖτο δὲ καὶ παρὰ τοῖς γεωμέτραις, τίνα ἄν τις τρόπον τὸ δοθὲν στερεὸν διαμένον ἐν τῷ αὐτῷ σχήματι διπλασιάσειεν, καὶ ἐκαλεῖτο τὸ τοιοῦτον πρόβλημα κύβου διπλασιασμός· ὑποθέμενοι γὰρ κύβον ἐζήτουν τοῦτον διπλασιάσαι. πάντων δὲ διαπορούντων ἐπὶ πολὺν χρόνον πρῶτος Ἱπποκράτης ὁ Χῖος ἐπενόησεν, ὅτι, ἐὰν εὑρεθῇ δύο εὐθειῶν γραμμῶν, ὧν ἡ μείζων τῆς ἐλάσσονός ἐστι διπλασία, δύο μέσας ἀνάλογον λαβεῖν ἐν συνεχεῖ ἀναλογίᾳ, διπλασιασθήσεται ὁ κύβος, ὥστε τὸ ἀπόρημα αὐτῷ εἰς ἕτερον οὐκ ἔλασσον ἀπόρημα κατέστρεφεν. μετὰ χρόνον δέ τινάς φασιν Δηλίους ἐπιβαλλομένους κατὰ χρησμὸν διπλασιάσαι τινὰ τῶν βωμῶν ἐμπεσεῖν εἰς τὸ αὐτὸ ἀπόρημα, διαπεμψαμένους δὲ τοὺς παρὰ τῷ Πλάτωνι ἐν Ἀκαδημίᾳ γεωμέτρας ἀξιοῦν αὐτοῖς εὑρεῖν τὸ ζητούμενον. τῶν δὲ φιλοπόνως ἐπιδιδόντων ἑαυτοὺς καὶ ζητούντων δύο τῶν δοθεισῶν δύο μέσας

[a] Valckenaer attributed these lines to Euripides, but Wilamowitz has shown that they cannot be from any play by Aeschylus, Sophocles or Euripides and must be the work of some minor poet.

[b] For if x, y are mean proportionals between a, b,

then $$\frac{a}{x}=\frac{x}{y}=\frac{y}{b}.$$

and as declaring, when he learnt it was a hundred feet each way: "Small indeed is the tomb thou hast chosen for a royal burial. Let it be double, and thou shalt not miss that fair form if thou quickly doublest each side of the tomb."[a] He seems to have made a mistake. For when the sides are doubled, the surface becomes four times as great and the solid eight times. It became a subject of inquiry among geometers in what manner one might double the given solid, while it remained the same shape, and this problem was called the duplication of the cube; for, given a cube, they sought to double it. When all were for a long time at a loss, Hippocrates of Chios first conceived that, if two mean proportionals could be found in continued proportion between two straight lines, of which the greater was double the lesser, the cube would be doubled,[b] so that the puzzle was by him turned into no less a puzzle. After a time, it is related, certain Delians, when attempting to double a certain altar in accordance with an oracle, fell into the same quandary, and sent over to ask the geometers who were with Plato in the Academy to find what they sought. When these men applied themselves diligently and sought to find two mean proportionals between two given straight lines,

Therefore $$y=\frac{x^2}{a}=\frac{ab}{x}$$

and, eliminating y, $$x^3=a^2b$$

so that $$\frac{a^3}{x^3}=\frac{a}{b}.$$

This property is stated in Eucl. *Elem.* v. Def. 10.

[b] If $b=2a$, then x is the side of a cube double a cube of side a. Once this was discovered by Hippocrates, the problem was always so treated.

λαβεῖν Ἀρχύτας μὲν ὁ Ταραντῖνος λέγεται διὰ τῶν ἡμικυλίνδρων εὑρηκέναι, Εὔδοξος δὲ διὰ τῶν καλουμένων καμπύλων γραμμῶν· συμβέβηκε δὲ πᾶσιν αὐτοῖς ἀποδεικτικῶς γεγραφέναι, χειρουργῆσαι δὲ καὶ εἰς χρείαν πεσεῖν μὴ δύνασθαι πλὴν ἐπὶ βραχύ τι τὸν Μέναιχμον καὶ ταῦτα δυσχερῶς. ἐπινενόηται δέ τις ὑφ' ἡμῶν ὀργανικὴ λῆψις ῥᾳδία, δι' ἧς εὑρήσομεν δύο τῶν δοθεισῶν οὐ μόνον δύο μέσας, ἀλλ' ὅσας ἄν τις ἐπιτάξῃ.

(b) Solutions given by Eutocius

Eutoc. *Comm. in Archim. de Sphaera et Cyl.* ii., Archim. ed. Heiberg iii. 54. 26–56. 12

Εἰς τὴν σύνθεσιν τοῦ α'

Τούτου ληφθέντος ἐπεὶ δι' ἀναλύσεως αὐτῷ προέβη τὰ τοῦ προβλήματος, ληξάσης τῆς ἀναλύσεως εἰς τὸ δεῖν δύο δοθεισῶν δύο μέσας ἀνάλογον προσευρεῖν ἐν συνεχεῖ ἀναλογίᾳ φησὶν ἐν τῇ συνθέσει· "εὑρήσθωσαν." τὴν δὲ εὕρεσιν τούτων ὑπ' αὐτοῦ μὲν γεγραμμένην οὐδὲ ὅλως εὑρίσκομεν, πολλῶν δὲ κλεινῶν ἀνδρῶν γραφαῖς ἐντετυχήκαμεν τὸ πρόβλημα τοῦτο ἐπαγγελλομέναις, ὧν τὴν Εὐδόξου τοῦ Κνιδίου παρῃτησάμεθα γραφήν, ἐπειδή φησιν μὲν ἐν προοιμίοις διὰ καμπύλων γραμμῶν αὐτὴν ηὑρηκέναι, ἐν δὲ τῇ ἀποδείξει πρὸς τῷ μὴ κεχρῆσθαι καμπύλαις γραμμαῖς ἀλλὰ καὶ

---

[a] "Given a cone or cylinder, to find a sphere equal to the cone or cylinder" (Archim. ed. Heiberg i. 170-174).

[b] This is a great misfortune, as we may be sure Eudoxus would have treated the subject in his usual brilliant fashion.

Archytas of Taras is said to have found them by the half-cylinders, and Eudoxus by the so-called curved lines ; but it turned out that all their solutions were theoretical, and they could not give a practical construction and turn it to use, except to a certain small extent Menaechmus, and that with difficulty. An easy mechanical solution was, however, found by me, and by means of it I will find, not only two means to the given straight lines, but as many as may be enjoined.

(b) Solutions given by Eutocius

Eutocius, *Commentary on Archimedes' Sphere and Cylinder* ii., Archim. ed. Heiberg iii. 54. 26–56. 12

On the Synthesis of Prop. 1 [a]

With this assumption the problem became for him one of analysis, and when the analysis resolved itself into the discovery of two mean proportionals in continuous proportion between two given straight lines he says in the synthesis : " Let them be found." How they were found we nowhere find described by him, but we have come across writings of many famous men dealing with this problem. Among them is Eudoxus of Cnidos, but we have omitted his account,[b] since he says in the preface that he made his discovery by means of curved lines, but in the demonstration itself not only did he not use curved

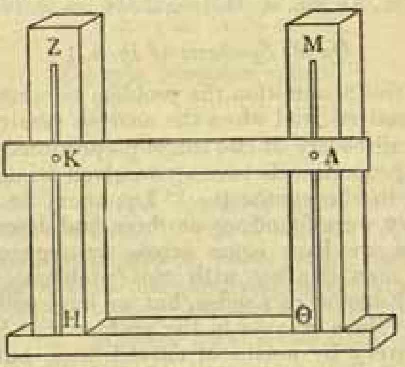
Tannery (*Mémoires scientifiques*, vol. i. pp. 53-61) suggests that Eudoxus's construction was a modified form of that by Archytas, for which see *infra*, pp. 284-289, the modification being virtually projection on the plane. Heath (*H.G.M.* i. 249-251) considers Tannery's suggestion ingenious and attractive, but too close an adaptation of Archytas's ideas to be the work or so original a mathematician as Eudoxus.

διῃρημένην ἀναλογίαν εὑρὼν ὡς συνεχεῖ χρῆται· ὅπερ ἦν ἄτοπον ὑπονοῆσαι, τί λέγω περὶ Εὐδόξου, ἀλλὰ περὶ τῶν καὶ μετρίως περὶ γεωμετρίαν ἀνεστραμμένων. ἵνα δὴ ἡ τῶν εἰς ἡμᾶς ἐληλυθότων ἀνδρῶν ἔννοια ἐμφανὴς γένηται, ὁ ἑκάστου τῆς εὑρέσεως τρόπος καὶ ἐνταῦθα γραφήσεται.

Ibid. 56. 13–58. 14

Ὡς Πλάτων

Δύο δοθεισῶν εὐθειῶν δύο μέσας ἀνάλογον εὑρεῖν ἐν συνεχεῖ ἀναλογίᾳ.



Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ ΑΒΓ πρὸς

[a] The complete list of solutions given by Eutocius is: Plato, Heron, Philon, Apollonius, Diocles, Pappus, Sporus, Menaechmus (two solutions), Archytas, Eratosthenes, Nicomedes.

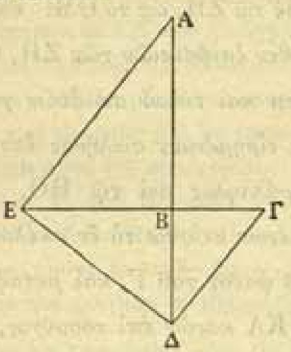
[b] It is virtually certain that this solution is wrongly attributed to Plato. Eutocius alone mentions it, and if it had been known to Eratosthenes he could hardly have failed to

lines but he used as continuous a discrete proportion which he found. That would be a foolish thing to imagine, not only of Eudoxus, but of any one moderately versed in geometry. In order that the ideas of those men who have come down to us may be made manifest, the manner in which each made his discovery will be described here also.[a]

Ibid. 56. 13–58. 14

(i.) *The Solution of Plato*[b]

Given two straight lines, to find two mean proportionals in continuous proportion.



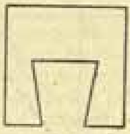
Let the two given straight lines be AB, BΓ, per-

cite it along with those of Archytas, Menaechmus and Eudoxus. Furthermore, Plato told the Delians, according to Plutarch's account, that Eudoxus or Helicon of Cyzicus would solve the problem for them: he did not apparently propose to tackle it himself. And Plutarch twice says that Plato objected to mechanical solutions as destroying the good of geometry, a statement which is consistent with his known attitude towards mathematics.

ὀρθὰς ἀλλήλαις, ὧν δεῖ δύο μέσας ἀνάλογον εὑρεῖν. ἐκβεβλήσθωσαν ἐπ' εὐθείας ἐπὶ τὰ Δ, Ε, καὶ κατεσκευάσθω ὀρθὴ γωνία ἡ ὑπὸ ΖΗΘ, καὶ ἐν ἑνὶ σκέλει, οἷον τῷ ΖΗ, κινείσθω κανὼν ὁ ΚΛ ἐν σωλῆνί τινι ὄντι ἐν τῷ ΖΗ οὕτως, ὥστε παράλληλον αὐτὸν διαμένειν τῷ ΗΘ. ἔσται δὲ τοῦτο, ἐὰν καὶ ἕτερον κανόνιον νοηθῇ συμφυὲς τῷ ΘΗ, παράλληλον δὲ τῷ ΖΗ, ὡς τὸ ΘΜ. σωληνισθεισῶν γὰρ τῶν ἄνωθεν ἐπιφανειῶν τῶν ΖΗ, ΘΜ σωλῆσιν πελεκινοειδέσιν καὶ τύλων συμφυῶν γενομένων τῷ ΚΛ εἰς τοὺς εἰρημένους σωλῆνας ἔσται ἡ κίνησις τοῦ ΚΛ παράλληλος ἀεὶ τῷ ΗΘ. τούτων οὖν κατεσκευασμένων κείσθω τὸ ἓν σκέλος τῆς γωνίας τυχὸν τὸ ΗΘ ψαῦον τοῦ Γ, καὶ μεταφερέσθω ἥ τε γωνία καὶ ὁ ΚΛ κανὼν ἐπὶ τοσοῦτον, ἄχρις ἂν τὸ μὲν Η σημεῖον ἐπὶ τῆς ΒΔ εὐθείας ᾖ τοῦ ΗΘ σκέλους ψαύοντος τοῦ Γ, ὁ δὲ ΚΛ κανὼν κατὰ μὲν τὸ Κ ψαύῃ τῆς ΒΕ εὐθείας, κατὰ δὲ τὸ λοιπὸν μέρος τοῦ Α, ὥστε εἶναι, ὡς ἔχει ἐπὶ τῆς καταγραφῆς, τὴν μὲν ὀρθὴν γωνίαν θέσιν ἔχουσαν ὡς τὴν ὑπὸ

pendicular to each other, between which it is required to find two mean proportionals. Let them be produced in a straight line to Δ, E, let the right-angle ZHΘ be constructed, and in one leg, say ZH, let the ruler KΛ be moved in a kind of groove in ZH, in such a way that it remains parallel to HΘ. This will come about if another ruler be conceived fixed to ΘH, but parallel to ZH, such as ΘM. If the upper surfaces of ZH, ΘM are grooved with axe-like grooves,[a] and there are notches on KΛ fitting into the aforementioned grooves, the motion of KΛ will always be parallel to HΘ. When this instrument is constructed, let one leg of the angle, say HΘ, be placed so as to touch Γ, and let the angle and the ruler KΛ be turned about until the point H falls upon the straight line BΔ, while the leg HΘ touches Γ, and the ruler KΛ touches the straight line BE at K, and in the other part touches A, so that it comes about, as in the figure, that the right angle takes up the position of the angle ΓΔE, while

[a] The grooves are presumably after the manner of the



accompanying diagram, or, as we should say, the notches and the grooves are *dove-tailed.*

ΓΔΕ, τὸν δὲ ΚΛ κανόνα θέσιν ἔχειν, οἵαν ἔχει ἡ ΕΑ· τούτων γὰρ γεναμένων ἔσται τὸ προκείμενον. ὀρθῶν γὰρ οὐσῶν τῶν πρὸς τοῖς Δ, Ε ἔστιν, ὡς ἡ ΓΒ πρὸς ΒΔ, ἡ ΔΒ πρὸς ΒΕ καὶ ἡ ΕΒ πρὸς ΒΑ.

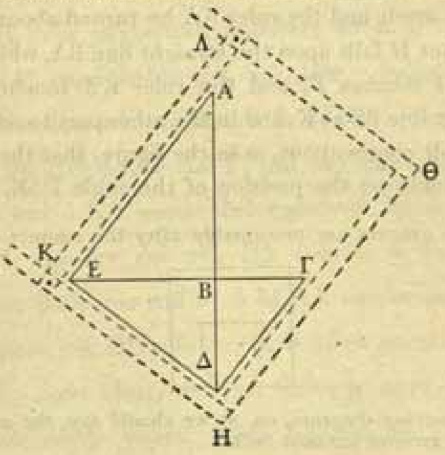
Ibid. 58. 15-16

Ὡς Ἥρων ἐν Μηχανικαῖς εἰσαγωγαῖς καὶ ἐν τοῖς Βελοποιικοῖς

Papp. *Coll.* iii. 9. 26, ed. Hultsch 62. 26–64. 18 ; Heron, *Mech.* i. 11, ed. Schmidt 268. 3–270. 15

Ἔστωσαν γὰρ αἱ δοθεῖσαι εὐθεῖαι αἱ ΑΒ, ΒΓ πρὸς ὀρθὰς ἀλλήλαις κείμεναι, ὧν δεῖ δύο μέσας ἀνάλογον εὑρεῖν.

[a] The account may become clearer from the accompanying diagram in which the instrument is indicated in its final



the ruler KΛ takes up the position EΛ.[a] When this is done, what was enjoined will be brought about. For since the angles at Δ, E are right, ΓB : BΔ = ΔB : BE = EB : BA. [Eucl. vi. 8, coroll.]

Ibid. 58. 15-16

(ii.) *The Solution of Heron in his "Mechanics" and "Construction of Engines of War"* [b]

Pappus, *Collection* iii. 9. 26, ed. Hultsch 62. 26–64. 18; Heron, *Mechanics* i. 11, ed. Schmidt 268. 3–270. 15

Let the two given straight lines between which it is required to find two mean proportionals be AB, BΓ lying at right angles one to another.

position by dotted lines. HΘ is made to pass through Γ and the instrument is turned until the point H lies on AB produced. The ruler is then moved until its edge KΛ passes through A. If K does not then lie on ΓB produced, the instrument has to be manipulated again until all conditions are fulfilled: (1) HΘ passes through Γ; (2) H lies on AB produced; (3) KΛ passes through A; (4) K lies on ΓB produced. It may not be easy to do this, but it is possible.

[b] Heron's own words have been most closely preserved by Pappus, whose version is here given in preference to Eutocius's, which includes some additions by the commentator. Schmidt also prefers Pappus's version in his edition of the Greek fragments of Heron's *Mechanics* in the Teubner edition of Heron's works (vol. ii., fasc. 1). The proof in the *Belopoeica* (edited by Wescher, *Poliorcétique des Grecs*, pp. 116-119) is extant. Philon of Byzantium and Apollonius gave substantially identical proofs.

Συμπεπληρώσθω τὸ ΑΒΓΔ παραλληλόγραμμον, καὶ ἐκβεβλήσθωσαν αἱ ΔΓ, ΔΑ, καὶ ἐπεζεύχθωσαν αἱ ΔΒ, ΓΑ, καὶ παρακείσθω κανόνιον πρὸς τῷ Β σημείῳ καὶ κινείσθω τέμνον τὰς ΓΕ, ΑΖ, ἄχρις οὗ ἡ ἀπὸ τοῦ Η ⟨ἀχθεῖσα⟩[1] ἐπὶ τὴν τῆς ΓΕ τομὴν ἴση γένηται τῇ ἀπὸ τοῦ Η ἐπὶ τὴν τῆς ΑΖ τομήν. γεγονέτω, καὶ ἔστω ἡ μὲν τοῦ κανονίου θέσις ἡ ΕΒΖ, ἴσαι δὲ αἱ ΕΗ, ΗΖ. λέγω οὖν ὅτι αἱ ΑΖ, ΓΕ μέσαι ἀνάλογόν εἰσιν τῶν ΑΒ, ΒΓ.

Ἐπεὶ γὰρ ὀρθογώνιόν ἐστιν τὸ ΑΒΓΔ παραλληλόγραμμον, αἱ τέσσαρες εὐθεῖαι αἱ ΔΗ, ΗΑ, ΗΒ, ΗΓ ἴσαι ἀλλήλαις εἰσίν. ἐπεὶ οὖν ἴση ἡ ΔΗ τῇ ΑΗ καὶ διῆκται ἡ ΗΖ, τὸ ἄρα ὑπὸ ΔΖΑ μετὰ τοῦ ἀπὸ ΑΗ ἴσον ἐστὶν τῷ ἀπὸ ΗΖ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ὑπὸ ΔΕΓ μετὰ τοῦ ἀπὸ ΓΗ ἴσον ἐστὶν τῷ ἀπὸ ΗΕ. καὶ εἰσὶν ἴσαι αἱ ΗΕ,

[1] ἀχθεῖσα add. Hultsch.

---

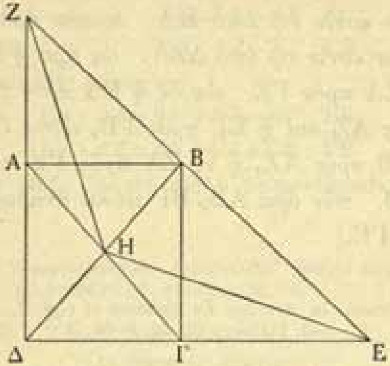
[a] The full proof requires ΗΘ to be drawn perpendicular to ΔΖ so that Θ bisects ΔΑ.

Then $\Delta Z \,.\, ZA + A\Theta^2 = Z\Theta^2$. [Eucl. ii. 6

Add $H\Theta^2$ to each side.

Then $\Delta Z \,.\, ZA + AH^2 = HZ^2$. [Eucl. i. 47

Let the parallelogram ΑΒΓΔ be completed, and let ΔΓ, ΔΑ be produced and let ΔΒ, ΓΑ be joined,



and let a ruler be placed at Β and moved about until the sections ΓΕ, ΑΖ cut off [from ΔΓ, ΔΑ produced] are such that the straight line drawn from Η to the section ΓΕ is equal to the straight line drawn from Η to the section ΑΖ. Let this be done, and let the position of the ruler be ΕΒΖ, so that ΕΗ, ΗΖ are equal. I say that ΑΖ, ΓΕ are mean proportionals between ΑΒ, ΒΓ.

For since the parallelogram ΑΒΓΔ is right-angled, the four straight lines ΔΗ, ΗΑ, ΗΒ, ΗΓ are equal one to another. Since ΔΗ is equal to ΑΗ, and ΗΖ has been drawn (from the vertex of the isosceles triangle ΑΗΔ to the base), therefore [a]

$$\Delta Z \,.\, ZA + AH^2 = HZ^2.$$

For the same reasons

$$\Delta E \,.\, E\Gamma + \Gamma H^2 = HE^2.$$

But ΗΕ, ΗΖ are equal.

ΗΖ. ἴσον ἄρα καὶ τὸ ὑπὸ ΔΖΑ μετὰ τοῦ ἀπὸ ΑΗ τῷ ὑπὸ ΔΕΓ μετὰ τοῦ ἀπὸ ΓΗ. ὧν τὸ ἀπὸ ΓΗ ἴσον ἐστὶν τῷ ἀπὸ ΗΑ. λοιπὸν ἄρα τὸ ὑπὸ ΔΕΓ ἴσον ἐστὶν τῷ ὑπὸ ΔΖΑ. ὡς ἄρα ἡ ΕΔ πρὸς ΔΖ, ἡ ΖΑ πρὸς ΓΕ. ὡς δὲ ἡ ΕΔ πρὸς ΔΖ, ἥ τε ΒΑ πρὸς ΑΖ καὶ ἡ ΕΓ πρὸς ΓΒ, ὥστε ἔσται καὶ ὡς ἡ ΑΒ πρὸς ΑΖ, ἥ τε ΖΑ πρὸς ΓΕ καὶ ἡ ΓΕ πρὸς ΓΒ. τῶν ἄρα ΑΒ, ΒΓ μέσαι ἀνάλογόν εἰσιν αἱ ΑΖ, ΓΕ.]

Eutoc. *Comm. in Archim. De Sphaera et Cyl.* ii., Archim. ed. Heiberg iii. 66. 8–70. 5

'Ως Διοκλῆς ἐν τῷ Περὶ πυρίων

'Εν κύκλῳ ἤχθωσαν δύο διάμετροι πρὸς ὀρθὰς αἱ ΑΒ, ΓΔ, καὶ δύο περιφέρειαι ἴσαι ἀπειλήφθωσαν ἐφ' ἑκάτερα τοῦ Β αἱ ΕΒ, ΒΖ, καὶ διὰ τοῦ Ζ παράλληλος τῇ ΑΒ ἤχθω ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΔΕ. λέγω, ὅτι τῶν ΓΗ, ΗΘ δύο μέσαι ἀνάλογόν εἰσιν αἱ ΖΗ, ΗΔ.

῎Ηχθω γὰρ διὰ τοῦ Ε τῇ ΑΒ παράλληλος ἡ

[a] Another fragment from the Περὶ πυρίων of Diocles is preserved by Eutocius (pp. 160 *et seq.*). It contains a solution by means of conics of the problem of dividing a sphere by a plane in such a way that the volumes of the resulting segments shall be in a given ratio, and refers both to Archi-

Therefore $\Delta Z . Z A + AH^2 = \Delta E . E\Gamma + \Gamma H^2$.

And $AH^2 = \Gamma H^2$.

Therefore $\Delta Z . ZA = \Delta E . E\Gamma$.

Therefore $E\Delta : \Delta Z = ZA : \Gamma E$.

But (by similar triangles)

$$E\Delta : \Delta Z = BA : AZ = E\Gamma : \Gamma B,$$

so that $AB : AZ = ZA : \Gamma E = \Gamma E : \Gamma B$.

Therefore AZ, ΓE are mean proportionals between AB, BΓ.]

Eutocius, *Commentary on Archimedes' Sphere and Cylinder* ii., Archim. ed. Heiberg iii. 66. 8–70. 5

(iii.) *The Solution of Diocles in his Book "On Burning Mirrors"* [a]

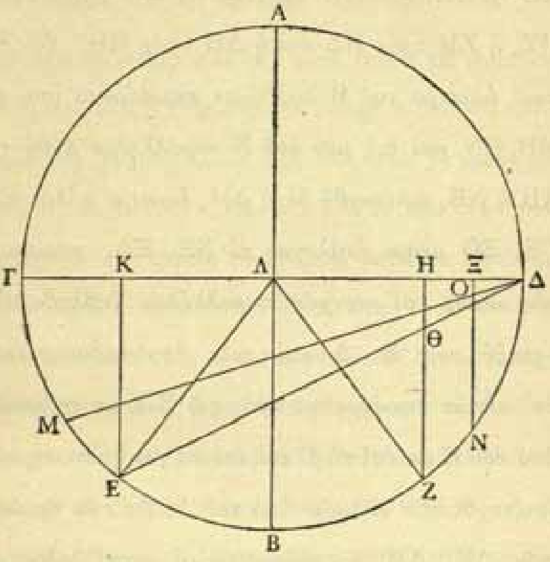
In a circle let there be drawn two diameters AB, ΓΔ at right angles, and on either side of B let there be cut off two equal arcs EB, BZ, and through Z let ZH be drawn parallel to AB, and let ΔE be joined. I say that ZH, HΔ are two mean proportionals between ΓH, HΘ.

For let EK be drawn through E parallel to AB;

medes and to Apollonius. Diocles must therefore have flourished later than these geometers. It appears also, from allusions in Proclus's commentary on Eucl. i., that the curve known to Geminus as the cissoid was none other than the curve here described and used by Diocles for finding two mean proportionals, though the identification is not certain (see Loria, *Le scienze esatte nell' antica Grecia*, pp. 410-415, Heath, *H.G.M.* i. 264). In that case, Diocles preceded Geminus, who flourished about 70 B.C. It is probable therefore that Diocles lived towards the end of the second century or the beginning of the first century B.C.

ΕΚ· ἴση ἄρα ἐστὶν ἡ μὲν ΕΚ τῇ ΖΗ, ἡ δὲ ΚΓ τῇ ΗΔ. ἔσται γὰρ τοῦτο δῆλον ἀπὸ τοῦ Λ ἐπὶ τὰ Ε, Ζ ἐπιζευχθεισῶν εὐθειῶν· ἴσαι γὰρ γίνονται αἱ ὑπὸ ΓΛΕ, ΖΛΔ, καὶ ὀρθαὶ αἱ πρὸς τοῖς Κ, Η· καὶ πάντα ἄρα πᾶσιν διὰ τὸ τὴν ΛΕ τῇ ΛΖ ἴσην εἶναι· καὶ λοιπὴ ἄρα ἡ ΓΚ τῇ ΗΔ ἴση ἐστίν. ἐπεὶ οὖν ἐστιν, ὡς ἡ ΔΚ πρὸς ΚΕ, ἡ ΔΗ πρὸς ΗΘ, ἀλλ' ὡς ἡ ΔΚ πρὸς ΚΕ, ἡ ΕΚ πρὸς ΚΓ· μέση γὰρ ἀνάλογον ἡ ΕΚ τῶν ΔΚ, ΚΓ· ὡς ἄρα ἡ ΔΚ πρὸς ΚΕ καὶ ἡ ΕΚ πρὸς ΚΓ, οὕτως ἡ ΔΗ

EK will therefore be equal to ZH, and KΓ to HΔ; this will be clear if straight lines are drawn joining



Λ to E, Z; for the angles ΓΛE, ZΛΔ are equal, and the angles at K, H are right; and therefore, since ΛE = ΛZ, all things will be equal to all; and therefore the remaining element ΓK is equal to HΔ. Now since

$$\Delta K : KE = \Delta H : H\Theta,$$

but $\Delta K : KE = EK : K\Gamma$ (for EK is a mean proportional between ΔK, KΓ),

therefore $\Delta K : KE = EK : K\Gamma = \Delta H : H\Theta.$

πρὸς ΗΘ. καί ἐστιν ἴση ἡ μὲν ΔΚ τῇ ΓΗ, ἡ δὲ ΚΕ τῇ ΖΗ, ἡ δὲ ΚΓ τῇ ΗΔ· ὡς ἄρα ἡ ΓΗ πρὸς ΗΖ, ἡ ΖΗ πρὸς ΗΔ καὶ ἡ ΔΗ πρὸς ΗΘ. ἐὰν δὴ παρ' ἑκάτερα τοῦ Β ληφθῶσιν περιφέρειαι ἴσαι αἱ ΜΒ, ΒΝ, καὶ διὰ μὲν τοῦ Ν παράλληλος ἀχθῇ τῇ ΑΒ ἡ ΝΞ, ἐπιζευχθῇ δὲ ἡ ΔΜ, ἔσονται πάλιν τῶν ΓΞ, ΞΟ μέσαι ἀνάλογον αἱ ΝΞ, ΞΔ. πλειόνων οὖν οὕτως καὶ συνεχῶν παραλλήλων ἐκβληθεισῶν μεταξὺ τῶν Β, Δ καὶ ταῖς ἀπολαμβανομέναις ὑπ' αὐτῶν περιφερείαις πρὸς τῷ Β ἴσων τεθεισῶν ἀπὸ τοῦ Β ὡς ἐπὶ τὸ Γ καὶ ἐπὶ τὰ γενάμενα σημεῖα ἐπιζευχθεισῶν εὐθειῶν ἀπὸ τοῦ Δ, ὡς τῶν ὁμοίων ταῖς ΔΕ, ΔΜ, τμηθήσονται αἱ παράλληλοι αἱ μεταξὺ τῶν Β, Δ κατά τινα σημεῖα, ἐπὶ τῆς προκειμένης καταγραφῆς τὰ Ο, Θ, ἐφ' ἃ κανόνος παραθέσει ἐπιζεύξαντες εὐθείας ἕξομεν καταγε-

And $\Delta K = \Gamma H$, $KE = ZH$, $K\Gamma = H\Delta$;

therefore $\Gamma H : HZ = ZH : H\Delta = \Delta H : H\Theta$.

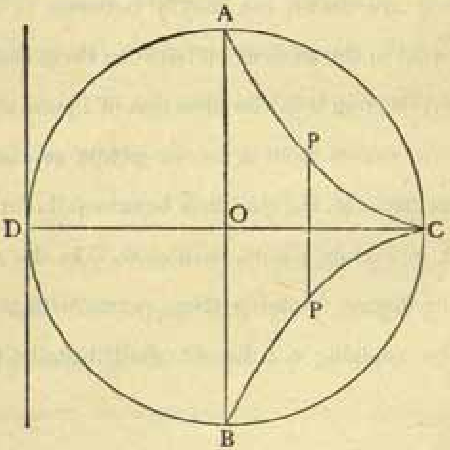
If then on either side of B there be cut off equal arcs MB, BN, and NΞ be drawn through N parallel to AB, and ΔM be joined, NΞ, ΞΔ, will again be mean proportionals between ΓΞ, ΞO. If in this way more parallels are drawn continually between B, Δ, and arcs equal to the arcs cut off between them and B are marked off from B in the direction of Γ, and straight lines are drawn from Δ to the points so obtained, such as ΔE, ΔM, the parallels between B and Δ will be cut in certain points, such as O, Θ in the accompanying figure. Joining these points with straight lines by applying a ruler we shall describe in the

γραμμένην ἐν τῷ κύκλῳ τινὰ γραμμήν, ἐφ' ἧς ἐὰν ληφθῇ τυχὸν σημεῖον καὶ δι' αὐτοῦ παράλληλος ἀχθῇ τῇ ΛΒ, ἔσται ἡ ἀχθεῖσα καὶ ἡ ἀπολαμβανομένη ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῷ Δ μέσαι ἀνάλογον τῆς τε ἀπολαμβανομένης ὑπ' αὐτῆς ἀπὸ τῆς διαμέτρου πρὸς τῷ Γ σημείῳ καὶ τοῦ μέρους αὐτῆς τοῦ ἀπὸ τοῦ ἐν τῇ γραμμῇ σημείου ἐπὶ τὴν ΓΔ διάμετρον.

Τούτων προκατεσκευασμένων ἔστωσαν αἱ δο-

---

[a] Lit. "line." It is noteworthy that Diocles, or Eutocius, conceived the curve as made up of an indefinite number of

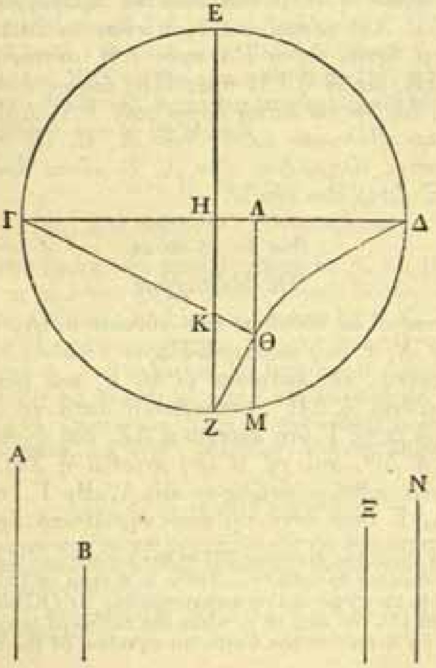


small straight lines, a typical Greek conception which has all the power of a theory of infinitesimals while avoiding its logical fallacies. The Greeks were never so modern as in this conception.

The curve described by Diocles has two branches, sym-

circle a certain curve,[a] and if on this any point be taken at random, and through it a straight line be drawn parallel to ΑΒ, the line so drawn and the portion of the diameter cut off by it in the direction of Δ will be mean proportionals between the portion of the diameter cut off by it in the direction of the point Γ and the part of the parallel itself between the point on the curve and the diameter ΓΔ.

With this preliminary construction, let the two



θεῖσαι δύο εὐθεῖαι, ὧν δεῖ δύο μέσας ἀνάλογον εὑρεῖν, αἱ Α, Β, καὶ ἔστω κύκλος, ἐν ᾧ δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΓΔ, ΕΖ, καὶ γεγράφθω ἐν αὐτῷ ἡ διὰ τῶν συνεχῶν σημείων γραμμή, ὡς προείρηται, ἡ ΔΘΖ, καὶ γεγονέτω, ὡς ἡ Α πρὸς τὴν Β, ἡ ΓΗ πρὸς ΗΚ, καὶ ἐπιζευχθεῖσα ἡ ΓΚ καὶ ἐκβληθεῖσα τεμνέτω τὴν γραμμὴν κατὰ τὸ Θ, καὶ διὰ τοῦ Θ τῇ ΕΖ παράλληλος ἤχθω ἡ ΛΜ· διὰ ἄρα τὰ προγεγραμμένα τῶν ΓΛ, ΛΘ μέσαι ἀνάλογόν εἰσιν αἱ ΜΛ, ΛΔ. καὶ ἐπεί ἐστιν, ὡς ἡ ΓΛ πρὸς ΛΘ, οὕτως ἡ ΓΗ πρὸς ΗΚ, ὡς δὲ ἡ ΓΗ πρὸς ΗΚ, οὕτως ἡ Α πρὸς τὴν Β, ἐὰν ἐν τῷ αὐτῷ λόγῳ ταῖς ΓΛ, ΛΜ, ΛΔ, ΛΘ παρεμβάλωμεν μέσας τῶν Α, Β, ὡς τὰς Ν, Ξ, ἔσονται εἰλημμέναι τῶν Α, Β μέσαι ἀνάλογον αἱ Ν, Ξ· ὅπερ ἔδει εὑρεῖν.

Ibid. 78. 13–80. 24

Ὡς Μέναιχμος

Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ Α, Ε· δεῖ δὴ τῶν Α, Ε δύο μέσας ἀνάλογον εὑρεῖν.

Γεγονέτω, καὶ ἔστωσαν αἱ Β, Γ, καὶ ἐκκείσθω θέσει εὐθεῖα ἡ ΔΗ πεπερασμένη κατὰ τὸ Δ, καὶ πρὸς τῷ Δ τῇ Γ ἴση κείσθω ἡ ΔΖ, καὶ ἤχθω πρὸς ὀρθὰς ἡ ΖΘ, καὶ τῇ Β ἴση κείσθω ἡ ΖΘ. ἐπεὶ οὖν τρεῖς εὐθεῖαι ἀνάλογον αἱ Α, Β, Γ, τὸ ὑπὸ τῶν Α, Γ ἴσον ἐστὶ τῷ ἀπὸ τῆς Β· τὸ ἄρα ὑπὸ

metrical about the diameter CD in the accompanying figure, and proceeding to infinity. There is a cusp at C and the tangent to the circle at D is an asymptote. If OC is the axis of x, and OA the axis of y, while the radius of the circle is a, then by definition the Cartesian equation of the curve is

given straight lines, between which it is required to find two mean proportionals, be Α, Β, and let there be a circle in which ΓΔ, ΕΖ are two diameters at right angles to each other, and let there be drawn in it through the successive points a curve ΔΘΖ, in the aforesaid manner, and let Α : Β = ΓΗ : ΗΚ, and let Γ, Κ be joined, and let the straight line joining them be produced so as to cut the line in Θ, and through Θ let ΛΜ be drawn parallel to ΕΖ ; therefore by what has been written previously ΜΛ, ΛΔ are mean proportionals between ΓΛ, ΛΘ. And since ΓΛ : ΛΘ = ΓΗ : ΗΚ and ΓΗ : ΗΚ = Α : Β, if between Α, Β we place means Ν, Ξ in the same ratio as ΓΛ, ΛΜ, ΛΔ, ΛΘ,[a] then Ν, Ξ will be mean proportionals between Α, Β ; which was to be found.

Ibid. 78. 13–80. 24

(iv.) *The Solutions of Menaechmus*

Let the two given straight lines be Α, Ε ; it is required to find two mean proportionals between Α, Ε.

Assume it done, and let the means be Β, Γ, and let there be placed in position a straight line ΔΗ, with an end point Δ, and at Δ let ΔΖ be placed equal to Γ, and let ΖΘ be drawn at right angles and let ΖΘ be equal to Β. Since the three straight lines Α, Β, Γ are in proportion, Α.Γ = Β^2 ; therefore the rectangle com-

$$\frac{a+x}{\sqrt{a^2-x^2}} = \frac{a-x}{y} \text{ or } y^2(a+x) = (a-x)^3.$$

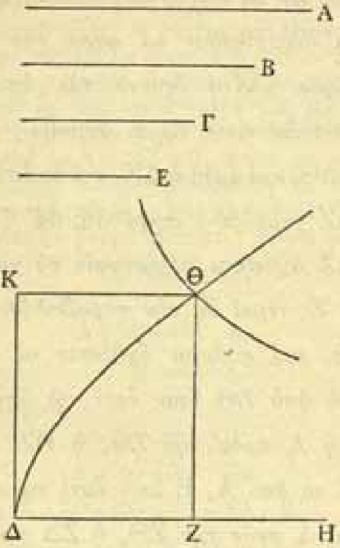
The curve was called by the Greeks the *cissoid* (κισσοειδὴς γραμμή) because the portion within the circle reminded them of a leaf of ivy (κισσός).

[a] *i.e.*, if we take ΓΛ : ΛΜ = Α : Ν, ΛΜ : ΛΔ = Ν : Ξ and ΛΔ : ΛΘ = Ξ : Β.

δοθείσης τῆς Α καὶ τῆς Γ, τουτέστι τῆς ΔΖ, ἴσον ἐστὶ τῷ ἀπὸ τῆς Β, τουτέστι τῷ ἀπὸ τῆς ΖΘ. ἐπὶ παραβολῆς ἄρα τὸ Θ διὰ τοῦ Δ γεγραμμένης. ἤχθωσαν παράλληλοι αἱ ΘΚ, ΔΚ. καὶ ἐπεὶ δοθὲν τὸ ὑπὸ Β, Γ—ἴσον γάρ ἐστι τῷ ὑπὸ Α, Ε—δοθὲν ἄρα καὶ τὸ ὑπὸ ΚΘΖ. ἐπὶ ὑπερβολῆς ἄρα τὸ Θ ἐν ἀσυμπτώτοις ταῖς ΚΔ, ΔΖ. δοθὲν ἄρα τὸ Θ· ὥστε καὶ τὸ Ζ.

Συντεθήσεται δὴ οὕτως. ἔστωσαν αἱ μὲν δοθεῖσαι εὐθεῖαι αἱ Α, Ε, ἡ δὲ τῇ θέσει ἡ ΔΗ πεπερασμένη κατὰ τὸ Δ, καὶ γεγράφθω διὰ τοῦ Δ

prehended by the given straight line A and the straight line Γ, that is, ΔZ, is equal to the square on



B, that is, to the square on ZΘ. Therefore Θ is on a parabola drawn through Δ. Let the parallels ΘK, ΔK be drawn. Then since the rectangle B . Γ is given —for it is equal to the rectangle A . E—the rectangle KΘ . ΘZ is given. The point Θ is therefore on a hyperbola with asymptotes KΔ, ΔZ. Therefore Θ is given; and so also is Z.

Let the synthesis be made in this manner. Let the given straight lines be A, E, let ΔH be a straight line given in position with an end point at Δ, and let

παραβολή, ἧς ἄξων μὲν ἡ ΔΗ, ὀρθία δὲ τοῦ εἴδους πλευρὰ ἡ Α, αἱ δὲ καταγόμεναι ἐπὶ τὴν ΔΗ ἐν ὀρθῇ γωνίᾳ δυνάσθωσαν τὰ παρὰ τὴν Α παρακείμενα χωρία πλάτη ἔχοντα τὰς ἀπολαμβανομένας ὑπ' αὐτῶν πρὸς τῷ Δ σημείῳ. γεγράφθω καὶ ἔστω ἡ ΔΘ, καὶ ὀρθὴ ἡ ΔΚ, καὶ ἐν ἀσυμπτώτοις ταῖς ΚΔ, ΔΖ γεγράφθω ὑπερβολή, ἀφ' ἧς αἱ παρὰ τὰς ΚΔ, ΔΖ ἀχθεῖσαι ποιήσουσιν τὸ χωρίον ἴσον τῷ ὑπὸ Α, Ε· τεμεῖ δὴ τὴν παραβολήν. τεμνέτω κατὰ τὸ Θ, καὶ κάθετοι ἤχθωσαν αἱ ΘΚ, ΘΖ. ἐπεὶ οὖν τὸ ἀπὸ ΖΘ ἴσον ἐστὶ τῷ ὑπὸ Α, ΔΖ, ἔστιν, ὡς ἡ Α πρὸς τὴν ΖΘ, ἡ ΘΖ πρὸς ΖΔ. πάλιν, ἐπεὶ τὸ ὑπὸ Α, Ε ἴσον ἐστὶ τῷ ὑπὸ ΘΖΔ, ἔστιν, ὡς ἡ Α πρὸς τὴν ΖΘ, ἡ ΖΔ πρὸς τὴν Ε. ἀλλ' ὡς ἡ Α πρὸς τὴν ΖΘ, ἡ ΖΘ πρὸς ΖΔ· καὶ ὡς ἄρα ἡ Α πρὸς τὴν ΖΘ, ἡ ΖΘ πρὸς ΖΔ καὶ ἡ ΖΔ πρὸς Ε. κείσθω τῇ μὲν ΘΖ ἴση ἡ Β, τῇ δὲ ΔΖ ἴση ἡ Γ. ἔστιν ἄρα, ὡς ἡ Α πρὸς τὴν Β, ἡ Β πρὸς τὴν Γ καὶ ἡ Γ πρὸς Ε. αἱ Α, Β, Γ, Ε ἄρα ἑξῆς ἀνάλογόν εἰσιν· ὅπερ ἔδει εὑρεῖν.

there be drawn through Δ a parabola whose axis is ΔH, and *latus rectum* A, and let the squares of the ordinates drawn at right angles to ΔH be equal to the areas applied to A having as their sides the straight lines cut off by them towards Δ. Let it be drawn, and let it be ΔΘ, and let ΔK be perpendicular [to ΔH], and in the asymptotes KΔ, ΔZ let there be drawn a hyperbola, such that the straight lines drawn parallel to KΔ, ΔZ will make an area equal to the rectangle comprehended by A, E. It will then cut the parabola. Let it cut at Θ, and let ΘK, ΘZ be drawn perpendicular. Since then

$$Z\Theta^2 = A \,.\, \Delta Z,$$

it follows that

$$A : Z\Theta = \Theta Z : Z\Delta.$$

Again, since $\quad A \,.\, E = \Theta Z \,.\, Z\Delta,$

it follows that

$$A : Z\Theta = Z\Delta : E.$$

But $\quad A : Z\Theta = Z\Theta : Z\Delta.$

Therefore $A : Z\Theta = Z\Theta : Z\Delta = Z\Delta : E.$

Let B be placed equal to ΘZ, and Γ equal to ΔZ. It follows that

$$A : B = B : \Gamma = \Gamma : E.$$

A, B, Γ, E are therefore in continuous proportion : which was to be found.[a]

[a] If a, x, y, b are in continuous proportion,

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}, \text{ and } x^2 = ay,\ y^2 = bx,\ xy = ab.$$

Therefore x, y may be determined as the intersection of the parabola $y^2 = bx$ and the hyperbola $xy = ab$. This is the

Ibid. 84. 12–88. 2

Ἡ Ἀρχύτου εὕρησις, ὡς Εὔδημος ἱστορεῖ

Ἔστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ ΑΔ, Γ· δεῖ δὴ τῶν ΑΔ, Γ δύο μέσας ἀνάλογον εὑρεῖν.

Γεγράφθω περὶ τὴν μείζονα τὴν ΑΔ κύκλος ὁ ΑΒΔΖ, καὶ τῇ Γ ἴση ἐνηρμόσθω ἡ ΑΒ καὶ ἐκβληθεῖσα συμπιπτέτω τῇ ἀπὸ τοῦ Δ ἐφαπτομένῃ τοῦ κύκλου κατὰ τὸ Π, παρὰ δὲ τὴν ΠΔΟ ἤχθω ἡ

---

analytical expression of the solution given above, where Ε $=a$ and Α $=b$. Menaechmus gave a second solution, reproduced by Eutocius, determining x, y as the intersection of the parabolas $x^2=ay$, $y^2=bx$.

This is the earliest known use of conic sections in the history of Greek mathematics, and Menaechmus is accordingly credited with their discovery. But the names parabola and hyperbola were not used by him; they are due to Apollonius; Menaechmus would have called them, with Archimedes, sections of a right-angled and obtuse-angled cone.

From the equations given above it follows that

$$x^2+y^2-bx-ay=0$$

is a circle passing through the points common to the parabolas

$$x^2=ay,\ y^2=bx.$$

It follows that x, y may be determined by the intersection of this circle with the hyperbola $xy=ab$.

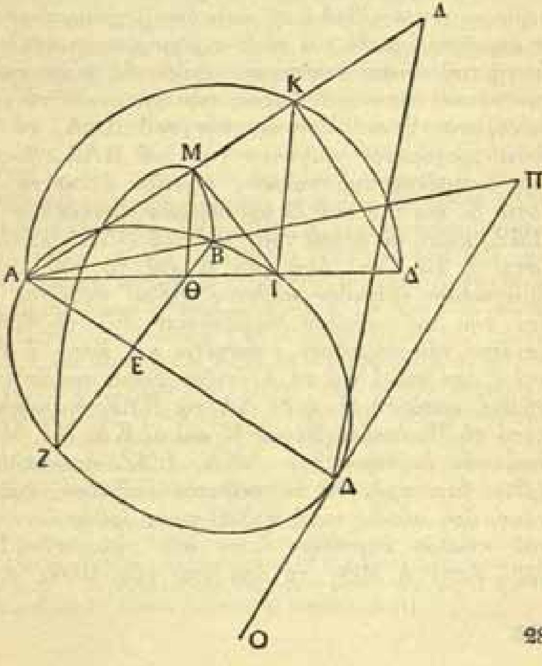
This is, in effect, the proof given by Heron, Philon and Apollonius. For, in the figure on p. 269, if ΔΖ, ΔΕ are the co-ordinate axes, ΑΒ $=a$, ΒΓ $=b$, then $x^2+y^2-bx-ay=0$ is the circle passing through Α, Β, Γ, and $xy=ab$ is the hyperbola having ΔΖ, ΔΕ as asymptotes and passing through Β.

Ibid. 84. 12–88. 2

(v.) *The Solution of Archytas, according to Eudemus*

Let the two given straight lines be AΔ, Γ; it is required to find two mean proportionals between AΔ, Γ.

Let the circle ABΔZ be described about the greater straight line AΔ, and let AB be inserted equal to Γ and let it be produced so as to meet at Π the tangent to the circle at Δ. Let BEZ be drawn parallel to ΠΔO,



ΒΕΖ, καὶ νενοήσθω ἡμικυλίνδριον ὀρθὸν ἐπὶ τοῦ ΑΒΔ ἡμικυκλίου, ἐπὶ δὲ τῆς ΑΔ ἡμικύκλιον ὀρθὸν ἐν τῷ τοῦ ἡμικυλινδρίου παραλληλογράμμῳ κείμενον· τοῦτο δὴ τὸ ἡμικύκλιον περιαγόμενον ὡς ἀπὸ τοῦ Δ ἐπὶ τὸ Β μένοντος τοῦ Α πέρατος τῆς διαμέτρου τεμεῖ τὴν κυλινδρικὴν ἐπιφάνειαν ἐν τῇ περιαγωγῇ καὶ γράψει ἐν αὐτῇ γραμμήν τινα. πάλιν δέ, ἐὰν τῆς ΑΔ μενούσης τὸ ΑΠΔ τρίγωνον περιενεχθῇ τὴν ἐναντίαν τῷ ἡμικυκλίῳ κίνησιν, κωνικὴν ποιήσει ἐπιφάνειαν τῇ ΑΠ εὐθείᾳ, ἣ δὴ περιαγομένη συμβαλεῖ τῇ κυλινδρικῇ γραμμῇ κατά τι σημεῖον· ἅμα δὲ καὶ τὸ Β περιγράψει ἡμικύκλιον ἐν τῇ τοῦ κώνου ἐπιφανείᾳ. ἐχέτω δὴ θέσιν κατὰ τὸν τόπον τῆς συμπτώσεως τῶν γραμμῶν τὸ μὲν κινούμενον ἡμικύκλιον ὡς τὴν τοῦ ΔΚΑ, τὸ δὲ ἀντιπεριαγόμενον τρίγωνον τὴν τοῦ ΔΛΑ, τὸ δὲ τῆς εἰρημένης συμπτώσεως σημεῖον ἔστω τὸ Κ, ἔστω δὲ καὶ διὰ τοῦ Β γραφόμενον ἡμικύκλιον τὸ ΒΜΖ, κοινὴ δὲ αὐτοῦ τομὴ καὶ τοῦ ΒΔΖΑ κύκλου ἔστω ἡ ΒΖ, καὶ ἀπὸ τοῦ Κ ἐπὶ τὸ τοῦ ΒΔΑ ἡμικυκλίου ἐπίπεδον κάθετος ἤχθω· πεσεῖται δὴ ἐπὶ τὴν τοῦ κύκλου περιφέρειαν διὰ τὸ ὀρθὸν ἑστάναι τὸν κύλινδρον. πιπτέτω καὶ ἔστω ἡ ΚΙ, καὶ ἡ ἀπὸ τοῦ Ι ἐπὶ τὸ Α ἐπιζευχθεῖσα συμβαλέτω τῇ ΒΖ κατὰ τὸ Θ, ἡ δὲ ΑΛ τῷ ΒΜΖ ἡμικυκλίῳ κατὰ τὸ Μ, ἐπεζεύχθωσαν δὲ καὶ αἱ ΚΔ, ΜΙ, ΜΘ. ἐπεὶ οὖν ἑκάτερον τῶν ΔΚΑ, ΒΜΖ ἡμικυκλίων ὀρθόν ἐστι πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ ἡ κοινὴ ἄρα αὐτῶν τομὴ ἡ ΜΘ πρὸς ὀρθάς ἐστι τῷ τοῦ κύκλου ἐπιπέδῳ· ὥστε καὶ πρὸς τὴν ΒΖ ὀρθή ἐστιν ἡ ΜΘ. τὸ ἄρα ὑπὸ τῶν ΒΘΖ, τοῦ-

and let a right half-cylinder be conceived upon the semicircle ABΔ, and on AΔ a right semicircle lying in the parallelogram of the half-cylinder. When this semicircle is moved about from Δ to B, the end point A of the diameter remaining fixed, it will cut the cylindrical surface in its motion and will describe in it a certain curve. Again, if AΔ be kept stationary and the triangle AΠΔ be moved about with an opposite motion to that of the semicircle, it will make a conic surface by means of the straight line AΠ, which in its motion will meet the curve on the cylinder in a certain point; at the same time B will describe a semicircle on the surface of the cone. Corresponding to the point in which the curves meet let the moving semicircle take up a position Δ′KA,[a] and the triangle moved in the opposite direction a position ΔΛA; let the point of the aforesaid meeting be K, and let BMZ be the semicircle described through B, and let BZ be the section common to it and the circle BΔZA, and let there be drawn from K a perpendicular upon the plane of the semicircle BΔA; it will fall upon the circumference of the circle because the cylinder is right. Let it fall, and let it be KI, and let the straight line joining I to A meet BZ in Θ; let AΛ meet the semicircle BMZ in M, and let KΔ, MI, MΘ be joined. Therefore since each of the semicircles Δ′KA, BMZ is at right angles to the underlying plane, their common section MΘ is also at right angles to the plane of the circle; so that MΘ is also at right angles to BZ. Therefore the rectangle contained by

[a] In the text and figure of the mss. the same letter is used to indicate the initial and final positions of Δ; for convenience they are distinguished in the figure and translation as Δ, Δ′. It would make the figure easier to grasp if Λ could be written Π′ (for Λ is the final position of Π).

τέστι τὸ ὑπὸ ΑΘΙ, ἴσον ἐστὶ τῷ ἀπὸ ΜΘ· ὅμοιον ἄρα ἐστὶ τὸ ΑΜΙ τρίγωνον ἑκατέρῳ τῶν ΜΙΘ, ΜΑΘ, καὶ ὀρθὴ ἡ ὑπὸ ΙΜΑ. ἔστιν δὲ καὶ ἡ ὑπὸ ΔΚΑ ὀρθή. παράλληλοι ἄρα εἰσὶν αἱ ΚΔ, ΜΙ, καὶ ἔσται ἀνάλογον, ὡς ἡ ΔΑ πρὸς ΑΚ, τουτέστιν ἡ ΚΑ πρὸς ΑΙ, οὕτως ἡ ΙΑ πρὸς ΑΜ, διὰ τὴν ὁμοιότητα τῶν τριγώνων. τέσσαρες ἄρα αἱ ΔΑ, ΑΚ, ΑΙ, ΑΜ ἑξῆς ἀνάλογόν εἰσιν. καί ἐστιν ἡ ΑΜ ἴση τῇ Γ, ἐπεὶ καὶ τῇ ΑΒ· δύο ἄρα δοθεισῶν τῶν ΑΔ, Γ δύο μέσαι ἀνάλογον ηὕρηνται αἱ ΑΚ, ΑΙ.

[a] The above solution is a remarkable achievement when it is remembered that Archytas flourished in the first half of the fourth century B.C., at which time Greek geometry was still in its infancy. It is quite easy, however, for us to represent the solution analytically. If ΑΔ is taken as the axis of x, the perpendicular to ΑΔ at Α in the plane of the paper as the axis of y, and the perpendicular to these lines as the axis of z, and if $\text{ΑΔ}=a$, $\Gamma=b$, then the point Κ is determined as the intersection of the following three curves:

(1) The cylinder $x^2+y^2=ax$,

(2) the curve formed by the motion of the half-circle about Α (a tore of inner diameter nil)

$$x^2+y^2+z^2=a\sqrt{x^2+y^2},$$

(3) the cone $x^2+y^2+z^2=\frac{a^2}{b^2}x^2$.

BΘ, ΘZ, which is the same as the rectangle contained by AΘ, ΘI, is equal to the square on MΘ ; therefore the triangle AMI is similar to each of the triangles MIΘ, MAΘ, and the angle IMA is right. The angle Δ′KA is also right. Therefore KΔ′, MI are parallel, and owing to the similarity of the triangles the following proportion holds :

$$\Delta' \mathrm{A} : \mathrm{AK} = \mathrm{KA} : \mathrm{AI} = \mathrm{IA} : \mathrm{AM}.$$

Therefore the four straight lines ΔA, AK, AI, AM are in continuous proportion. And AM is equal to Γ, since it is equal to AB ; therefore to the two given straight lines AΔ, Γ, two mean proportionals, AK, AI, have been found.[a]

Since K is the point of intersection,

$$\mathrm{AK} = \sqrt{x^2+y^2+z^2}, \ \mathrm{AI} = \sqrt{x^2+y^2}.$$

From (2) it follows directly that

$$\mathrm{AK}^2 = a \,.\, \mathrm{AI}$$

i.e.,

$$\frac{a}{\mathrm{AK}} = \frac{\mathrm{AK}}{\mathrm{AI}}.$$

From (1) and (3) it follows that

$$x^2+y^2+z^2 = \frac{(x^2+y^2)^2}{b^2}$$

$$\therefore \ \sqrt{x^2+y^2+z^2} = \frac{x^2+y^2}{b}$$

i.e.,

$$\mathrm{AK} = \frac{\mathrm{AI}^2}{b}$$

or

$$\frac{\mathrm{AK}}{\mathrm{AI}} = \frac{\mathrm{AI}}{b}$$

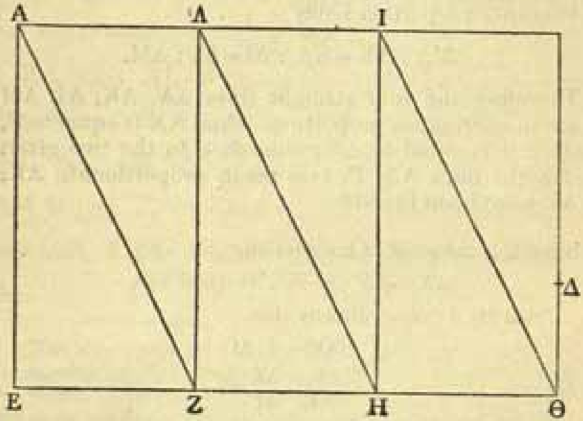
$$\therefore \ \frac{a}{\mathrm{AK}} = \frac{\mathrm{AK}}{\mathrm{AI}} = \frac{\mathrm{AI}}{b},$$

and AK, AI are mean proportionals between a and b.

Ibid. 88. 3–96. 27

Ὡς Ἐρατοσθένης . . .

Δεδόσθωσαν δύο ἄνισοι εὐθεῖαι, ὧν δεῖ δύο μέσας ἀνάλογον εὑρεῖν ἐν συνεχεῖ ἀναλογίᾳ, αἱ ΑΕ, ΔΘ, καὶ κείσθω ἐπί τινος εὐθείας τῆς ΕΘ



πρὸς ὀρθὰς ἡ ΑΕ, καὶ ἐπὶ τῆς ΕΘ τρία συνεστάτω παραλληλόγραμμα ἐφεξῆς τὰ ΑΖ, ΖΙ, ΙΘ, καὶ ἤχθωσαν διάμετροι ἐν αὐτοῖς αἱ ΑΖ, ΛΗ, ΙΘ· ἔσονται δὴ αὗται παράλληλοι. μένοντος δὴ τοῦ μέσου παραλληλογράμμου τοῦ ΖΙ συνωσθήτω τὸ μὲν ΑΖ ἐπάνω τοῦ μέσου, τὸ δὲ ΙΘ ὑποκάτω, καθάπερ ἐπὶ τοῦ δευτέρου σχήματος, ἕως οὗ γένηται τὰ Α, Β, Γ, Δ κατ' εὐθεῖαν, καὶ διήχθω διὰ τῶν Α, Β, Γ, Δ σημείων εὐθεῖα καὶ συμπιπτέτω τῇ ΕΘ ἐκβληθείσῃ κατὰ τὸ Κ· ἔσται δή, ὡς ἡ ΑΚ πρὸς ΚΒ, ἐν μὲν ταῖς ΑΕ, ΖΒ παραλ-

Ibid. 88. 3–96. 27 [a]

(vi.) *The Solution of Eratosthenes* . . .

Let there be given two unequal straight lines AE, ΔΘ between which it is required to find two mean proportionals in continued proportion, and let AE be placed at right angles to the straight line EΘ, and upon EΘ let there be erected three successive parallelograms [b] AZ, ZI, IΘ, and let the diagonals AZ, ΛH, IΘ be drawn therein; these will be parallel. While the middle parallelogram ZI remains stationary, let the other two approach each other, AZ above the middle one, IΘ below it, as in the second figure,[c] until A, B, Γ, Δ lie along a straight line, and let a straight line be drawn through the points A, B, Γ, Δ, and let it meet EΘ produced in K; it will follow that in the parallels AE, ZB

$$AK : KB = EK : KZ$$

[a] This is the letter falsely purporting to be by Eratosthenes of which the beginning has already been cited, *supra*, pp. 256-261. The extract here given (δεδόσθωσαν . . .) starts in Heiberg's text at 90. 30. Eratosthenes' solution is given, with variations, by Pappus, *Collection* iii. 7, ed. Hultsch 56. 18–58. 22.

[b] Pappus says triangles in his account; it makes no difference.

[c] See p. 294.

λήλοις ἡ ΕΚ πρὸς ΚΖ, ἐν δὲ ταῖς ΑΖ, ΒΗ παραλλήλοις ἡ ΖΚ πρὸς ΚΗ. ὡς ἄρα ἡ ΑΚ πρὸς ΚΒ, ἡ ΕΚ πρὸς ΚΖ καὶ ἡ ΚΖ πρὸς ΚΗ. πάλιν, ἐπεί ἐστιν, ὡς ἡ ΒΚ πρὸς ΚΓ, ἐν μὲν ταῖς ΒΖ, ΓΗ παραλλήλοις ἡ ΖΚ πρὸς ΚΗ, ἐν δὲ ταῖς ΒΗ, ΓΘ παραλλήλοις ἡ ΗΚ πρὸς ΚΘ, ὡς ἄρα ἡ ΒΚ πρὸς ΚΓ, ἡ ΖΚ πρὸς ΚΗ καὶ ἡ ΗΚ πρὸς ΚΘ. ἀλλ' ὡς ἡ ΖΚ πρὸς ΚΗ, ἡ ΕΚ πρὸς ΚΖ· καὶ ὡς ἄρα ἡ ΕΚ πρὸς ΚΖ, ἡ ΖΚ πρὸς ΚΗ καὶ ἡ ΗΚ πρὸς ΚΘ. ἀλλ' ὡς ἡ ΕΚ πρὸς ΚΖ, ἡ ΑΕ πρὸς ΒΖ, ὡς δὲ ἡ ΖΚ πρὸς ΚΗ, ἡ ΒΖ πρὸς ΓΗ, ὡς δὲ ἡ ΗΚ πρὸς ΚΘ, ἡ ΓΗ πρὸς ΔΘ· καὶ ὡς ἄρα ἡ ΑΕ πρὸς ΒΖ, ἡ ΒΖ πρὸς ΓΗ καὶ ἡ ΓΗ πρὸς ΔΘ. ηὕρηνται ἄρα τῶν ΑΕ, ΔΘ δύο μέσαι ἥ τε ΒΖ καὶ ἡ ΓΗ.

Ταῦτα οὖν ἐπὶ τῶν γεωμετρουμένων ἐπιφανειῶν ἀποδέδεικται· ἵνα δὲ καὶ ὀργανικῶς δυνώμεθα τὰς δύο μέσας λαμβάνειν, διαπήγνυται πλινθίον ξύλινον ἢ ἐλεφάντινον ἢ χαλκοῦν ἔχον τρεῖς πινακίσκους ἴσους ὡς λεπτοτάτους, ὧν ὁ μὲν μέσος ἐνήρμοσται, οἱ δὲ δύο ἐπωστοί εἰσιν ἐν χολέδραις, τοῖς δὲ μεγέθεσιν καὶ ταῖς συμμετρίαις ὡς ἕκαστοι ἑαυτοὺς πείθουσιν· τὰ μὲν γὰρ τῆς ἀποδείξεως ὡσαύτως συντελεῖται· πρὸς δὲ τὸ ἀκριβέστερον λαμβάνεσθαι τὰς γραμμὰς φιλοτεχνητέον, ἵνα ἐν τῷ συνάγεσθαι τοὺς πινακίσκους παράλληλα διαμένῃ πάντα καὶ ἄσχαστα καὶ ὁμαλῶς συναπτόμενα ἀλλήλοις.

Ἐν δὲ τῷ ἀναθήματι τὸ μὲν ὀργανικὸν χαλκοῦν ἐστιν καὶ καθήρμοσται ὑπ' αὐτὴν τὴν στεφάνην τῆς στήλης προσμεμολυβδοχοημένον, ὑπ' αὐτοῦ δὲ ἡ ἀπόδειξις συντομώτερον φραζομένη καὶ τὸ σχῆμα,

and in the parallels AZ, BH

$$AK : KB = ZK : KH.$$

Therefore $\quad AK : KB = EK : KZ = KZ : KH.$

Again, since in the parallels BZ, ΓH

$$BK : K\Gamma = ZK : KH$$

and in the parallels BH, ΓΘ

$$BK : K\Gamma = HK : K\Theta,$$

therefore $\quad BK : K\Gamma = ZK : KH = HK : K\Theta.$

But $ZK : KH = EK : KZ$, and therefore

$$EK : KZ = ZK : KH = HK : K\Theta.$$

But $EK : KZ = AE : BZ$, $ZK : KH = BZ : \Gamma H$,

$$HK : K\Theta = \Gamma H : \Delta\Theta.$$

Therefore $\quad AE : BZ = BZ : \Gamma H = \Gamma H : \Delta\Theta.$

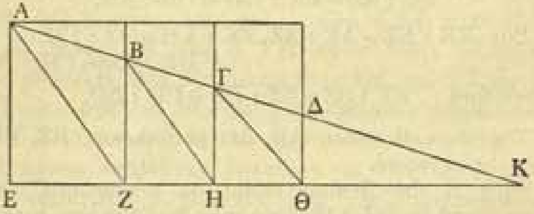
Therefore between AE, ΔΘ two means, BZ, ΓH, have been found.

Such is the demonstration on geometrical surfaces; and in order that we may find the two means mechanically, a board of wood or ivory or bronze is pierced through, having on it three equal tablets, as smooth as possible, of which the midmost is fixed and the two outside run in grooves, their sizes and proportions being a matter of individual choice—for the proof is accomplished in the same manner; in order that the lines may be found with the greatest accuracy, the instrument must be skilfully made, so that when the tablets are moved everything remains parallel, smoothly fitting without a gap.

In the votive gift the instrument is of bronze and is fastened on with lead close under the crown of the pillar, and beneath it is a shortened form of the proof

μετ' αὐτὸ δὲ ἐπίγραμμα. ὑπογεγράφθω οὖν σοι καὶ ταῦτα, ἵνα ἔχῃς καὶ ὡς ἐν τῷ ἀναθήματι. τῶν δὲ δύο σχημάτων τὸ δεύτερον γέγραπται ἐν τῇ στήλῃ.

"Δύο τῶν δοθεισῶν εὐθειῶν δύο μέσας ἀνάλογον εὑρεῖν ἐν συνεχεῖ ἀναλογίᾳ. δεδόσθωσαν αἱ ΑΕ, ΔΘ. συνάγω δὴ τοὺς ἐν τῷ ὀργάνῳ πίνακας, ἕως ἂν κατ' εὐθεῖαν γένηται τὰ Α, Β, Γ, Δ σημεῖα. νοείσθω δή, ὡς ἔχει ἐπὶ τοῦ δευτέρου σχήματος. ἔστιν ἄρα, ὡς ἡ ΑΚ πρὸς ΚΒ, ἐν μὲν ταῖς ΑΕ, ΒΖ παραλλήλοις ἡ ΕΚ πρὸς ΚΖ, ἐν δὲ ταῖς ΑΖ, ΒΗ ἡ ΖΚ πρὸς ΚΗ· ὡς ἄρα ἡ ΕΚ πρὸς ΚΖ, ἡ



ΚΖ πρὸς ΚΗ. ὡς δὲ αὗται πρὸς ἀλλήλας, ἥ τε ΑΕ πρὸς ΒΖ καὶ ἡ ΒΖ πρὸς ΓΗ. ὡσαύτως δὲ δείξομεν, ὅτι καί, ὡς ἡ ΖΒ πρὸς ΓΗ, ἡ ΓΗ πρὸς ΔΘ· ἀνάλογον ἄρα αἱ ΑΕ, ΒΖ, ΓΗ, ΔΘ. ηὕρηνται ἄρα δύο τῶν δοθεισῶν δύο μέσαι.

"Ἐὰν δὲ αἱ δοθεῖσαι μὴ ἴσαι ὦσιν ταῖς ΑΕ, ΔΘ, ποιήσαντες αὐταῖς ἀνάλογον τὰς ΑΕ, ΔΘ τούτων ληψόμεθα τὰς μέσας καὶ ἐπανοίσομεν ἐπ' ἐκείνας, καὶ ἐσόμεθα πεποιηκότες τὸ ἐπιταχθέν. ἐὰν δὲ πλείους μέσας ἐπιταχθῇ εὑρεῖν, ἀεὶ ἑνὶ πλείους πινακίσκους καταστησόμεθα ἐν τῷ ὀργανίῳ τῶν ληφθησομένων μέσων· ἡ δὲ ἀπόδειξις ἡ αὐτή·

and the figure, and along with this is an epigram. These also shall be written below for you, in order that you may have what is on the votive gift. Of the two figures, the second is that which is inscribed on the pillar.[a]

"Between two given straight lines to find two means in continuous proportion. Let AE, ΔΘ be the given straight lines. Then I move the tables in the instrument until the points A, B, Γ, Δ are in the same straight line. Let this be pictured as in the second figure. Then AK : KB is equal, in the parallels AE, BZ, to EK : KZ, and in the parallels AZ, BH to ZK : KH; therefore EK : KZ = KZ : KH. Now this is also the ratio AE : BZ and BZ : ΓH. Similarly we shall show that ZB : ΓH = ΓH : ΔΘ; AE, BZ, ΓH, ΔΘ are therefore proportional. Between the two given straight lines two means have therefore been found.

"If the given straight lines are not equal to AE, ΔΘ, by making AE, ΔΘ proportional to them and taking the means between these and then going back to the original lines, we shall do what was enjoined. If it is required to find more means, we shall continually insert more tables in the instrument according to the number of means to be taken; and the proof is the same.

[a] The short proof and epigram which follow are presumably the genuine work of Eratosthenes, being taken from the votive gift. The reference to the *second* figure cannot, however, be genuine as there was only one figure on the votive offering; perhaps δεύτερον should be omitted.

"Εἰ κύβον ἐξ ὀλίγου διπλήσιον, ὤγαθε, τεύχειν
  φράζεαι ἢ στερεὴν πᾶσαν ἐς ἄλλο φύσιν
εὖ μεταμορφῶσαι, τόδε τοι πάρα, κἂν σύ γε
    μάνδρην
  ἢ σιρὸν ἢ κοίλου φρείατος εὐρὺ κύτος
τῇδ' ἀναμετρήσαιο, μέσας ὅτε τέρμασιν ἄκροις
  συνδρομάδας δισσῶν ἐντὸς ἕλῃς κανόνων.
μηδὲ σύ γ' Ἀρχύτεω δυσμήχανα ἔργα κυλίνδρων
  μηδὲ Μεναιχμείους κωνοτομεῖν τριάδας
διζήσῃ, μηδ' εἴ τι θεουδέος Εὐδόξοιο
  καμπύλον ἐγ γραμμαῖς εἶδος ἀναγράφεται.
τοῖσδε γὰρ ἐν πινάκεσσι μεσόγραφα μυρία τεύχοις
  ῥεῖά κεν ἐκ παύρου πυθμένος ἀρχόμενος.
εὐαίων, Πτολεμαῖε, πατὴρ ὅτι παιδὶ συνηβῶν
  πάνθ', ὅσα καὶ Μούσαις καὶ βασιλεῦσι φίλα,
αὐτὸς ἐδωρήσω· τὸ δ' ἐς ὕστερον, οὐράνιε Ζεῦ,
  καὶ σκήπτρων ἐκ σῆς ἀντιάσειε χερός.
καὶ τὰ μὲν ὣς τελέοιτο, λέγοι δέ τις ἄνθεμα λεύσ-
    σων
  τοῦ Κυρηναίου τοῦτ' Ἐρατοσθένεος."

Ibid. 98. 1-7

'Ως Νικομήδης ἐν τῷ Περὶ κογχοειδῶν γραμμῶν

Γράφει δὲ καὶ Νικομήδης ἐν τῷ ἐπιγεγραμμένῳ πρὸς αὐτοῦ Περὶ κογχοειδῶν συγγράμματι ὀργάνου κατασκευὴν τὴν αὐτὴν ἀποπληροῦντος χρείαν, ἐφ' ᾧ καὶ μεγάλα μὲν σεμνυνόμενος φαίνεται ὁ ἀνήρ, πολλὰ δὲ τοῖς Ἐρατοσθένους ἐπεγγελῶν εὑρήμασιν

---

[a] Or "with a small effort," Heiberg.

[b] Perhaps so called because there are three conic sections—of an acute-angled, right-angled and obtuse-angled cone

"If, good friend, thou thinkest to produce from a small [cube] [a] one double thereof, or duly to change any solid figure into another nature, this is in thy power, and thou canst measure a byre or corn-pit or the broad basin of a hollow well by this method, when thou takest between two rulers means converging with their extreme ends. Do not seek to do the difficult business of the cylinders of Archytas, or to cut the cone in the triads [b] of Menaechmus, or to produce any such curved form in lines as is described by the divine Eudoxus. Indeed, on these tablets thou couldst easily find a thousand means, beginning from a small base. Happy art thou, O Ptolemy, a father who lives his son's life in all things, in that thou hast given him such things as are dear to the Muses and kings; and in the future, O heavenly Zeus, may he also receive the sceptre from thy hands. May this prayer be fulfilled, and may anyone seeing this votive offering say: This is the gift of Eratosthenes of Cyrene."

Ibid. 98. 1-7

(vii.) *The Solution of Nicomedes in his Book "On Conchoidal Lines"* [c]

Nicomedes also describes, in the book written by him *On Conchoids*, the construction of an instrument fulfilling the same purpose, upon which it appears he prided himself exceedingly, greatly deriding the

(ellipse, parabola and hyperbola). If so, this proves that Menaechmus discovered the ellipse as well as the other two.

[c] It follows from this extract that Nicomedes was later than Eratosthenes; and as Apollonius called a certain curve "sister of the *cochloid*" (*infra*, p. 334), he must have been younger than Apollonius. He was therefore born about 270 B.C.

ὡς ἀμηχάνοις τε ἅμα καὶ γεωμετρικῆς ἕξεως ἐστερημένοις.

Papp. *Coll.* iv. 26. 39–28. 43, ed. Hultsch 242. 13–250. 25

κϛʹ. Εἰς τὸν διπλασιασμὸν τοῦ κύβου παράγεταί τις ὑπὸ Νικομήδους γραμμὴ καὶ γένεσιν ἔχει τοιαύτην.

Ἐκκείσθω εὐθεῖα ἡ ΑΒ, καὶ αὐτῇ πρὸς ὀρθὰς ἡ ΓΔΖ, καὶ εἰλήφθω τι σημεῖον ἐπὶ τῆς ΓΔΖ δοθὲν τὸ Ε, καὶ μένοντος τοῦ Ε σημείου ἐν ᾧ ἐστιν τόπῳ ἡ ΓΔΕΖ εὐθεῖα φερέσθω κατὰ τῆς ΑΔΒ εὐθείας ἑλκομένη διὰ τοῦ Ε σημείου οὕτως ὥστε διὰ παντὸς φέρεσθαι τὸ Δ ἐπὶ τῆς ΑΒ εὐθείας καὶ μὴ ἐκπίπτειν ἑλκομένης τῆς ΓΔΕΖ διὰ τοῦ Ε. τοιαύτης δὴ κινήσεως γενομένης ἐφ᾽ ἑκάτερα φανερὸν ὅτι τὸ Γ σημεῖον γράψει γραμμὴν οἵα ἐστὶν ἡ ΛΓΜ, καὶ ἔστιν αὐτῆς τὸ σύμπτωμα τοιοῦτον. ὡς ἂν εὐθεῖα προσπίπτῃ τις ἀπὸ τοῦ Ε σημείου πρὸς τὴν γραμμήν, τὴν ἀπολαμβανομένην μεταξὺ τῆς τε ΑΒ εὐθείας καὶ τῆς ΛΓΜ γραμμῆς ἴσην εἶναι τῇ ΓΔ εὐθείᾳ· μενούσης γὰρ τῆς ΑΒ καὶ μένοντος τοῦ Ε σημείου, ὅταν γένηται τὸ Δ ἐπὶ τὸ Η, ἡ ΓΔ εὐθεῖα τῇ ΗΘ ἐφαρμόσει καὶ τὸ Γ σημεῖον ἐπὶ τὸ Θ ⟨πεσεῖται⟩[1]· ἴση ἄρα ἐστὶν ἡ ΓΔ τῇ ΗΘ. ὁμοίως καὶ ἐὰν ἑτέρα τις

[1] πεσεῖται add. Hultsch.

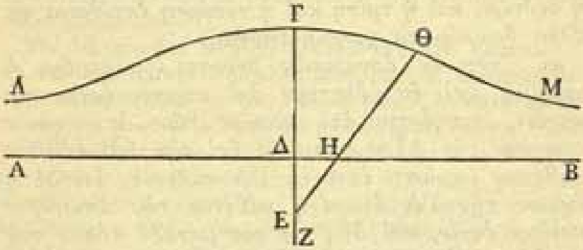
[a] Eutocius proceeds to describe Nicomedes' solution; we shall give an alternative account by Pappus.

discoveries of Eratosthenes as impracticable and lacking in geometrical sense.[a]

Pappus, *Collection* iv. 26. 39–28. 43, ed. Hultsch
242. 13–250. 25

26. For the duplication of the cube a certain line is drawn by Nicomedes and generated in this way.

Let there be a straight line AB, with ΓΔZ at right angles to it, and on ΓΔZ let there be taken a certain



given point E, and while the point E remains in the same position let the straight line ΓΔEZ be drawn through the point E and moved about the straight line AΔB in such a way that Δ always moves along the straight line AB and does not fall beyond it while ΓΔEZ is drawn through E. The motion being after this fashion on either side, it is clear that the point Γ will describe a curve such as ΛΓM, and its property is of this nature: when any straight line drawn from the point E falls upon the curve, the portion cut off between the straight line AB and the curve ΛΓM is equal to the straight line ΓΔ; for AB is stationary and the point E fixed, and when Δ goes to H, the straight line ΓΔ will coincide with HΘ and the point Γ will fall upon Θ; therefore ΓΔ is equal to HΘ.

ἀπὸ τοῦ Ε σημείου πρὸς τὴν γραμμὴν προσπέσῃ, τὴν ἀποτεμνομένην ὑπὸ τῆς γραμμῆς καὶ τῆς ΑΒ εὐθείας ἴσην ποιήσει τῇ ΓΔ [ἐπειδὴ ταύτῃ ἴσαι εἰσὶν αἱ προσπίπτουσαι].[1] καλείσθω δέ, φησίν, ἡ μὲν ΑΒ εὐθεῖα κανών, τὸ δὲ σημεῖον πόλος, διάστημα δὲ ἡ ΓΔ, ἐπειδὴ ταύτῃ ἴσαι εἰσὶν αἱ προσπίπτουσαι πρὸς τὴν ΛΓΜ γραμμήν, αὐτὴ δὲ ἡ ΛΓΜ γραμμὴ κοχλοειδὴς πρώτη (ἐπειδὴ καὶ ἡ δευτέρα καὶ ἡ τρίτη καὶ ἡ τετάρτη ἐκτίθεται εἰς ἄλλα θεωρήματα χρησιμεύουσαι).

κζʹ. Ὅτι δὲ ὀργανικῶς δύναται γράφεσθαι ἡ γραμμὴ καὶ ἐπ' ἔλαττον ἀεὶ συμπορεύεται τῷ κανόνι, τουτέστιν ὅτι πασῶν τῶν ἀπό τινων σημείων τῆς ΛΓΘ γραμμῆς ἐπὶ τὴν ΑΒ εὐθεῖαν καθέτων μεγίστη ἐστὶν ἡ ΓΔ κάθετος, ἀεὶ δὲ ἡ ἔγγιον τῆς ΓΔ ἀγομένη κάθετος τῆς ἀπώτερον μείζων ἐστίν, καὶ ὅτι, εἰς τὸν μεταξὺ τόπον τοῦ κανόνος καὶ τῆς κοχλοειδοῦς ἐάν τις ᾖ εὐθεῖα, ἐκβαλλομένη τμηθήσεται ὑπὸ τῆς κοχλοειδοῦς, αὐτὸς ἀπέδειξεν ὁ Νικομήδης, καὶ ἡμεῖς ἐν τῷ εἰς τὸ Ἀνάλημμα Διοδώρου, τρίχα τεμεῖν τὴν γωνίαν βουλόμενοι, κεχρήμεθα τῇ προειρημένῃ γραμμῇ.

[1] ἐπειδὴ . . . προσπίπτουσαι "ex proximis inepte huc translata" del. Hultsch.

---

[a] Let a be the interval or constant intercept between the curve and the base, and b the distance from the pole to the base (ΕΔ). If Θ is any point on the curve, and $E\Theta = r$, $\angle \Gamma E\Theta = \phi$, then the fundamental equation of the curve is

$$r = b \sec \phi + a.$$

If a is measured *backwards* from the base towards the pole, then another conchoidal figure is obtained on the same side of the base as the pole, having for its fundamental equation

$$r = b \sec \phi - a.$$

This takes three forms according as a is greater than,

Similarly, if any other straight line drawn from the point E falls upon the curve, the portion cut off by the curve and the straight line AB will make a straight line equal to ΓΔ. Now, says he, let the straight line AB be called the ruler, the point [E] the pole, ΓΔ the interval, since the straight lines falling upon the line ΛΓΜ are equal to it, and let the curve ΛΓΜ itself be called the first cochloidal line (since there are second and third and fourth cochloids which are useful for other theorems).[a]

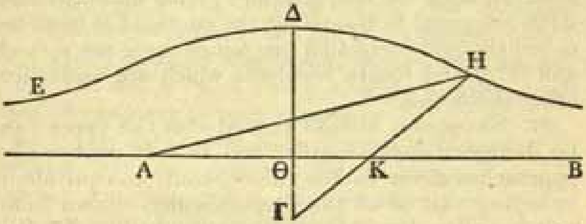
27. Nicomedes himself proved that the curve can be described mechanically, and that it continually approaches closer to the ruler—which is equivalent to saying that of all the perpendiculars drawn from points on the line ΛΓΘ to the straight line AB the greatest is the perpendicular ΓΔ, while the perpendicular drawn nearer to ΓΔ is always greater than the more remote; he also proved that any straight line in the space between the ruler and the cochloid will be cut, when produced, by the cochloid; and we used the aforesaid line in the commentary on the *Analemma*[b] of Diodorus when we sought to trisect an angle.

equal to, or less than b. These three forms are probably the "second, third and fourth cochloids," but we have no direct information. When a is greater than b, the curve has a loop at the pole; when a equals b, there is a cusp at the pole; when a is less than b, there is no double point.

The original name of the curve would appear to be the *cochloid* (κοχλοειδὴς γραμμή), as it is called by Pappus, from a supposed resemblance to a shell-fish (κόχλος). Later it was called the *conchoid* (κογχοειδὴς γραμμή), the "mussel-like" curve.

[b] Diodorus of Alexandria lived in the time of Caesar and is commemorated in the *Anthology* (xiv. 139) as a maker of gnomons. Ptolemy also wrote an *Analemma*, whose object is a graphic representation on a plane of parts of the heavenly sphere.

Διὰ δὴ τῶν εἰρημένων φανερὸν ὡς δυνατόν ἐστιν γωνίας δοθείσης ὡς τῆς ὑπὸ ΗΑΒ καὶ σημείου ἐκτὸς αὐτῆς τοῦ Γ διάγειν τὴν ΓΗ καὶ ποιεῖν τὴν ΚΗ μεταξὺ τῆς γραμμῆς καὶ τῆς ΑΒ ἴσην τῇ δοθείσῃ.



Ἤχθω κάθετος ἀπὸ τοῦ Γ σημείου ἐπὶ τὴν ΑΒ ἡ ΓΘ καὶ ἐκβεβλήσθω, καὶ τῇ δοθείσῃ ἴση ἔστω ἡ ΔΘ, καὶ πόλῳ μὲν τῷ Γ, διαστήματι δὲ τῷ δοθέντι, τουτέστιν τῇ ΔΘ, κανόνι δὲ τῷ ΑΒ γεγράφθω κοχλοειδὴς γραμμὴ πρώτη ἡ ΕΔΗ· συμβάλλει ἄρα τῇ ΑΗ διὰ τὸ προλεχθέν. συμβαλλέτω κατὰ τὸ Η, καὶ ἐπεζεύχθω ἡ ΓΗ· ἴση ἄρα καὶ ἡ ΚΗ τῇ δοθείσῃ.

κη΄. Τινὲς δὲ τῆς χρήσεως ἕνεκα παρατιθέντες κανόνα τῷ Γ κινοῦσιν αὐτόν, ἕως ἂν ἐκ τῆς πείρας ἡ μεταξὺ ἀπολαμβανομένη τῆς ΑΒ εὐθείας καὶ τῆς ΕΔΗ γραμμῆς ἴση γένηται τῇ δοθείσῃ· τούτου γὰρ ὄντος τὸ προκείμενον ἐξ ἀρχῆς δείκνυται (λέγω δὲ κύβος κύβου διπλάσιος εὑρίσκεται). πρότερον δὲ δύο δοθεισῶν εὐθειῶν δύο μέσαι κατὰ τὸ συνεχὲς ἀνάλογον λαμβάνονται, ὧν ὁ μὲν Νικομήδης τὴν κατασκευὴν ἐξέθετο μόνον, ἡμεῖς

Now by what has been said it is clear that if there is an angle, such as HAB, and a point Γ outside the angle, it is possible so to draw ΓH as to make KH between the line and AB equal to a given straight line.

Let ΓΘ be drawn from the point Γ perpendicular to AB and produced to Δ so that ΔΘ is equal to the given straight line, and with Γ for pole, the given straight line, that is ΔΘ, for interval, and AB for ruler let the first cochloid EΔH be drawn; then by what has been said above it will meet AH; let it meet it in H, and let ΓH be joined; KH will therefore be equal to the given straight line.

28. Some people, following [a more convenient] usage, apply a ruler to Γ and move it until by trial the portion between the straight line AB and the line EΔH becomes equal to the given straight line; and when this is done the problem which was posed at the outset is solved (I mean a cube which is double of a cube is found). But first two means in continuous proportion are taken between two given straight lines; Nicomedes explained only the construction necessary

δὲ καὶ τὴν ἀπόδειξιν ἐφηρμόσαμεν τῇ κατασκευῇ τὸν τρόπον τοῦτον.

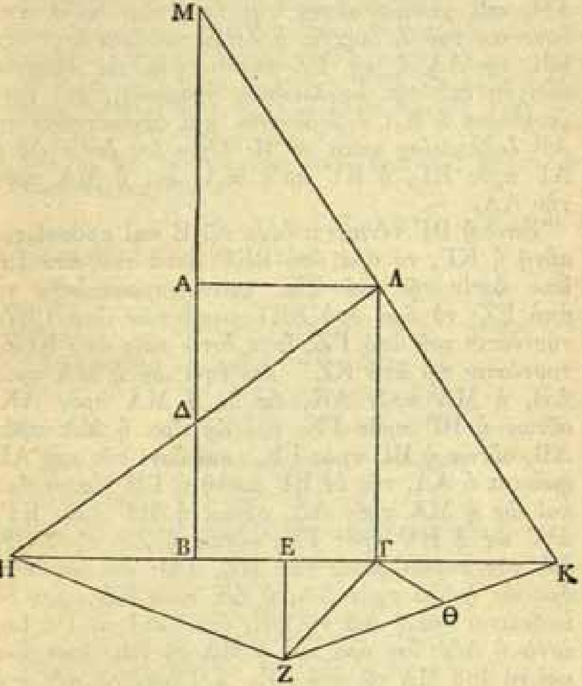
Δεδόσθωσαν γὰρ δύο εὐθεῖαι αἱ ΓΛ, ΛΑ πρὸς ὀρθὰς ἀλλήλαις, ὧν δεῖ δύο μέσας ἀνάλογον κατὰ τὸ συνεχὲς εὑρεῖν, καὶ συμπεπληρώσθω τὸ ΑΒΓΛ παραλληλόγραμμον, καὶ τετμήσθω δίχα ἑκατέρα τῶν ΑΒ, ΒΓ τοῖς Δ, Ε σημείοις, καὶ ἐπιζευχθεῖσα

---

[a] The proof is given by Eutocius with very few variations (pp. 104–106) and also in another place by Pappus himself (iii. 8, ed. Hultsch 58. 23–62. 13, with several differences). In iii. 8 the straight lines are called ΔΓ, ΔΑ, whereas here and in the passage from Eutocius the mss. have ΓΛ, ΛΑ. Wherever we have Λ here, it is reasonably certain that Pappus wrote Δ, and *vice versa*.

for doing this, but we have supplied a proof to the construction in this manner.

Let[a] there be given two straight lines ΓΛ, ΛΑ at right angles to each other between which it is required



to find two means in continuous proportion, and let the parallelogram ΑΒΓΛ be completed, and let each of the straight lines ΑΒ, ΒΓ be bisected at the points

μὲν ἡ ΔΛ ἐκβεβλήσθω καὶ συμπιπτέτω τῇ ΓΒ ἐκβληθείσῃ κατὰ τὸ Η, τῇ δὲ ΒΓ πρὸς ὀρθὰς ἡ ΕΖ, καὶ προσβεβλήσθω ἡ ΓΖ ἴση οὖσα τῇ ΑΔ, καὶ ἐπεζεύχθω ἡ ΖΗ καὶ αὐτῇ παράλληλος ἡ ΓΘ, καὶ γωνίας οὔσης τῆς ὑπὸ τῶν ΚΓΘ ἀπὸ δοθέντος τοῦ Ζ διήχθω ἡ ΖΘΚ ποιοῦσα ἴσην τὴν ΘΚ τῇ ΑΔ ἢ τῇ ΓΖ (τοῦτο γὰρ ὡς δυνατὸν ἐδείχθη διὰ τῆς κοχλοειδοῦς γραμμῆς), καὶ ἐπιζευχθεῖσα ἡ ΚΛ ἐκβεβλήσθω καὶ συμπιπτέτω τῇ ΑΒ ἐκβληθείσῃ κατὰ τὸ Μ· λέγω ὅτι ἐστὶν ὡς ἡ ΛΓ πρὸς ΚΓ, ἡ ΚΓ πρὸς ΜΑ, καὶ ἡ ΜΑ πρὸς τὴν ΑΛ.

Ἐπεὶ ἡ ΒΓ τέτμηται δίχα τῷ Ε καὶ πρόσκειται αὐτῇ ἡ ΚΓ, τὸ ἄρα ὑπὸ ΒΚΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσον ἐστὶν τῷ ἀπὸ ΕΚ. κοινὸν προσκείσθω τὸ ἀπὸ ΕΖ· τὸ ἄρα ὑπὸ ΒΚΓ μετὰ τῶν ἀπὸ ΓΕΖ, τουτέστιν τοῦ ἀπὸ ΓΖ, ἴσον ἐστὶν τοῖς ἀπὸ ΚΕΖ, τουτέστιν τῷ ἀπὸ ΚΖ. καὶ ἐπεὶ ὡς ἡ ΜΑ πρὸς ΑΒ, ἡ ΜΛ πρὸς ΛΚ, ὡς δὲ ἡ ΜΛ πρὸς ΛΚ, οὕτως ἡ ΒΓ πρὸς ΓΚ, καὶ ὡς ἄρα ἡ ΜΑ πρὸς ΑΒ, οὕτως ἡ ΒΓ πρὸς ΓΚ. καὶ ἔστι τῆς μὲν ΑΒ ἡμίσεια ἡ ΑΔ, τῆς δὲ ΒΓ διπλῆ ἡ ΓΗ· ἔσται ἄρα καὶ ὡς ἡ ΜΑ πρὸς ΑΔ, οὕτως ἡ ΗΓ πρὸς ΚΓ. ἀλλ' ὡς ἡ ΗΓ πρὸς ΓΚ, οὕτως ἡ ΖΘ πρὸς ΘΚ διὰ τὰς παραλλήλους τὰς ΗΖ, ΓΘ· καὶ συνθέντι ἄρα ὡς ἡ ΜΔ πρὸς ΔΑ, ἡ ΖΚ πρὸς ΚΘ. ἴση δὲ ὑπόκειται καὶ ἡ ΑΔ τῇ ΘΚ, ἐπεὶ[1] καὶ τῇ ΓΖ ἴση ἐστὶν ἡ ΑΔ[1]· ἴση ἄρα καὶ ἡ ΜΔ τῇ ΖΚ· ἴσον ἄρα καὶ τὸ ἀπὸ ΜΔ τῷ ἀπὸ ΖΚ. καὶ ἔστι τῷ μὲν ἀπὸ ΜΔ ἴσον τὸ ὑπὸ ΒΜΑ μετὰ τοῦ ἀπὸ ΔΑ, τῷ δὲ

[1] ἐπεὶ . . . ΑΔ. Hultsch thinks these words are interpolated; they appear in both other versions.

Δ, E respectively, and let $\Delta\Lambda$ be joined and produced, and let it meet ΓB produced in H, and let EZ be drawn at right angles to BΓ in such a way that ΓZ is equal to AΔ, and let ZH be joined and parallel to it let $\Gamma\Theta$ be drawn, and, since the angle K$\Gamma\Theta$ is given, from the given point Z let ZΘK be so drawn as to make ΘK equal to AΔ or to ΓZ (that this is possible is proved by the cochloidal line), and let KΛ be joined and produced, and let it meet AB produced in M; I say that $\Lambda\Gamma : K\Gamma = K\Gamma : M\Lambda = M\Lambda : A\Lambda$.

Since BΓ is bisected at E and KΓ lies in BΓ produced, therefore

$$BK \cdot K\Gamma + \Gamma E^2 = EK^2. \qquad \text{[Eucl. ii. 6}$$

Let EZ^2 be added to both sides.

Therefore $\quad BK \cdot K\Gamma + \Gamma E^2 + EZ^2 = EK^2 + EZ^2,$

that is $\quad BK \cdot K\Gamma + \Gamma Z^2 = KZ^2. \qquad$ [Eucl. i. 47

And since $\quad MA : AB = M\Lambda : \Lambda K$

and $\quad M\Lambda : \Lambda K = B\Gamma : \Gamma K,$

therefore $\quad MA : AB = B\Gamma : \Gamma K.$

And $\quad A\Delta = \frac{1}{2}AB, \ \Gamma H = 2B\Gamma.$

Therefore $\quad MA : A\Delta = H\Gamma : K\Gamma.$

But on account of HZ, $\Gamma\Theta$ being parallels,

$$H\Gamma : \Gamma K = Z\Theta : \Theta K.$$

Therefore, compounding,

$$M\Delta : \Delta A = ZK : K\Theta.$$

But by hypothesis $\quad A\Delta = \Theta K$, since $\Gamma Z = A\Delta$;

therefore $\quad M\Delta = ZK;$

therefore $\quad M\Delta^2 = ZK^2.$

And $\quad M\Delta^2 = BM \cdot MA + \Delta A^2 \qquad$ [Eucl. ii. 6

ἀπὸ ΖΚ ἴσον ἐδείχθη τὸ ὑπὸ ΒΚΓ μετὰ τοῦ ἀπὸ ΖΓ, ὧν τὸ ἀπὸ ΑΔ ἴσον τῷ ἀπὸ ΓΖ (ἴση γὰρ ὑπόκειται ἡ ΑΔ τῇ ΓΖ)· ἴσον ἄρα καὶ τὸ ὑπὸ ΒΜΑ τῷ ὑπὸ ΒΚΓ· ὡς ἄρα ἡ ΜΒ πρὸς ΒΚ, ἡ ΓΚ πρὸς ΜΑ. ἀλλ' ὡς ἡ ΒΜ πρὸς ΒΚ, ἡ ΛΓ πρὸς ΓΚ· ὡς ἄρα ἡ ΛΓ πρὸς ΓΚ, ἡ ΓΚ πρὸς ΑΜ. ἔστι δὲ καὶ ὡς ἡ ΜΒ πρὸς ΒΚ, ἡ ΜΑ πρὸς ΑΛ· καὶ ὡς ἄρα ἡ ΛΓ πρὸς ΓΚ, ἡ ΓΚ πρὸς ΑΜ, καὶ ἡ ΑΜ πρὸς ΑΛ.

2. SQUARING OF THE CIRCLE

(a) General

Plut. *De Exil.* 17, 607E, F

Ἀνθρώπου δ' οὐδεὶς ἀφαιρεῖται τόπος εὐδαιμονίαν, ὥσπερ οὐδ' ἀρετὴν οὐδὲ φρόνησιν. ἀλλ' Ἀναξαγόρας μὲν ἐν τῷ δεσμωτηρίῳ τὸν τοῦ κύκλου τετραγωνισμὸν ἔγραφε.

Aristoph. *Aves* 1001-1005

ΜΕΤΩΝ. Προσθεὶς οὖν ἐγὼ
τὸν κανόν' ἄνωθεν τουτονὶ τὸν καμπύλον,
ἐνθεὶς διαβήτην—μανθάνεις; ΠΕΙΣΘΕΤΑΙΡΟΣ.
οὐ μανθάνω.
ΜΕΤΩΝ. Ὀρθῷ μετρήσω κανόνι προστιθείς, ἵνα
ὁ κύκλος γένηταί σοι τετράγωνος.

---

[a] This reference shows the popularity of the problem of squaring the circle in 414 B.C., when the *Birds* was first produced. Meton, who is here burlesqued, is the great astronomer who about eighteen years earlier had found that after any period of 6940 days (a little over nineteen solar

and it was proved that

$$ZK^2 = BK \,.\, K\Gamma + Z\Gamma^2,$$

and here $\Gamma Z^2 = A\Delta^2$ (for by hypothesis $A\Delta = \Gamma Z$);

therefore $BM \,.\, MA = BK \,.\, K\Gamma$;

therefore $MB : BK = \Gamma K : MA$. [Eucl. vi. 16

But $BM : BK = A\Gamma : \Gamma K$;

therefore $A\Gamma : \Gamma K = \Gamma K : AM$.

And $MB : BK = MA : A\Lambda$;

and therefore $A\Gamma : \Gamma K = \Gamma K : AM = AM : A\Lambda$.

2. SQUARING OF THE CIRCLE

(a) General

Plutarch, *On Exile* 17, 607E, F

There is no place that can take away the happiness of a man, nor yet his virtue or wisdom. Anaxagoras, indeed, wrote on the squaring of the circle while in the prison.

Aristophanes, *Birds* 1001-1005 [a]

Meton. So then applying here my flexible rod, and there my compass—you understand? Peisthetairos. I don't.

Meton. With the straight rod I measure so that the circle may become a square for you.

years) the sun and moon occupy the same relative positions as at the beginning, and had just built a water-clock worked by water from a neighbouring spring on the Colonus in the Athenian Agora. Actually, Meton made no contribution to squaring the circle: all he seems to be represented as doing is to divide the circle into four quadrants by two diameters at right angles.

(b) Approximation by Polygons

(i.) *Antiphon*

Aristot. *Phys.* A 2, 185 a 14-17

Ἅμα δ' οὐδὲ λύειν ἅπαντα προσήκει, ἀλλ' ἢ ὅσα ἐκ τῶν ἀρχῶν τις ἐπιδεικνὺς ψεύδεται, ὅσα δὲ μή, οὔ, οἷον τὸν τετραγωνισμὸν τὸν μὲν διὰ τῶν τμημάτων γεωμετρικοῦ διαλῦσαι, τὸν δὲ Ἀντιφῶντος οὐ γεωμετρικοῦ.

Them. *in Phys.* A 2 (Aristot. 185 a 14), ed. Schenkl 3. 30–4. 7

Ἐπεὶ καὶ τὰ ψευδογραφήματα ὅσα μὲν σῴζει τὰς γεωμετρικὰς ὑποθέσεις λυτέον τῷ γεωμέτρῃ, ὅσα δὲ μάχεται πρὸς ἐκείνας, παραιτητέον, οἷον

---

[a] Antiphon was an Athenian sophist contemporary with Socrates.

[b] The comments of Themistius, Philoponus and Simplicius on this passage are of great importance in the history of Greek geometry. All three agree (Simplicius with a reservation) that "the quadrature by means of segments" is to be ascribed to Hippocrates of Chios. Simplicius's reproduction of the passage in Eudemus's *History of Geometry* which tells us of certain areas squared by Hippocrates has already been given (*supra*, pp. 234-253). The four quadratures there given contain no fallacy. What then is the fallacy with which Aristotle and the commentators charge Hippocrates? It is most probably an alleged assumption by Hippocrates that because he had squared a particular lune in each of three kinds, he had squared all types of lunes; and, as he had also squared a figure consisting of a lune and a circle, that he had squared the circle. In fact, the last-mentioned lune was not of a kind which he had previously squared, and so he had not really squared the circle. But did Hippocrates think that he had squared the circle? There is no reason to suppose that he so thought, and it is extremely unlikely that a mathematician of his calibre

(b) Approximation by Polygons

(i.) *Antiphon* [a]

Aristotle, *Physics* A 2, 185 a 14-17

At the same time it is not convenient to refute everything, but only false demonstrations starting from the fundamental principles, and otherwise not; thus it is the business of the geometer to refute the quadrature by means of segments, but it is not the business of the geometer to refute that of Antiphon.[b]

Themistius, *Commentary on Aristotle's Physics* A 2 (185 a 14), ed. Schenkl 3. 30–4. 7

For such false arguments as preserve the geometrical hypotheses are to be refuted by geometry, but such as conflict with them are to be left alone.

could be so deluded. Heiberg (*Philol.* xliii. 336-344) thinks that in the then state of logic he may have thought he had squared the circle. Björnbo (in Pauly-Wissowa, *Real-Encyclopädie*, xvi. 1787–1799) thinks he knew perfectly well what he had done, but used language calculated to give the impression that he had squared the circle. Both suggestions are highly improbable. Heath (*H.G.M.* i. 197) prefers to think that Hippocrates was trying to put what he had discovered in the most favourable light. Ross (*Aristotle's Physics*, p. 466) is of opinion that Hippocrates simply proved his quadratures of lunes and the sum of a lune and circle, no doubt in the hope of ultimately squaring the circle, but without any claim to have done so. This appears the best view. Aristotle has misunderstood what Hippocrates claimed to have done.

τμήματα means "segments," and is not properly used of "lunes," but mathematical terminology was fluid in Aristotle's time, and τμῆμα may have been used to denote any portion cut out of a circle. In *De Caelo* ii. 8, 290 a 4, Aristotle uses it to denote a "sector."

δύο τινὲς κύκλον ἐπιχειρήσαντες τετραγωνίζειν Ἱπποκράτης τε ὁ Χῖος καὶ ὁ Ἀντιφῶν. τὸν μὲν οὖν Ἱπποκράτους λυτέον. τὰς γὰρ ἀρχὰς φυλάττων παραλογίζεται τῷ μόνον μὲν ἐκεῖνον τὸν μηνίσκον τετραγωνίσαι ὃς γράφεται περὶ τὴν τοῦ τετραγώνου πλευρὰν τοῦ εἰς τὸν κύκλον ἐγγραφομένου, πάντα[1] δὲ μηνίσκον οἷόν τε τετραγωνίζειν λαβεῖν εἰς[1] ἀπόδειξιν, πρὸς Ἀντιφῶντα δὲ οὐκέτ' ἂν ἔχοι λέγειν ὁ γεωμέτρης, ὃς ἐγγράφων τρίγωνον ἰσόπλευρον εἰς τὸν κύκλον καὶ ἐφ' ἑκάστης τῶν πλευρῶν ἕτερον ἰσοσκελὲς συνιστὰς πρὸς τῇ περιφερείᾳ τοῦ κύκλου καὶ τοῦτο ἐφεξῆς ποιῶν ᾤετό ποτε ἐφαρμόσειν τοῦ τελευταίου τριγώνου τὴν πλευρὰν εὐθεῖαν οὖσαν τῇ περιφερείᾳ.

Simpl. *in Phys.* A 2 (Aristot. 185 a 14), ed. Diels 54. 20–55. 24

[a] Ὁ δὲ Ἀντιφῶν γράψας κύκλον ἐνέγραψέ τι χωρίον εἰς αὐτὸν πολύγωνον τῶν ἐγγράφεσθαι δυναμένων. ἔστω δὲ εἰ τύχοι τετράγωνον τὸ ἐγγεγραμμένον. . . . καὶ δῆλον ὅτι ἡ συναγωγὴ παρὰ τὰς γεωμετρικὰς ἀρχὰς γέγονεν οὐχ ὡς ὁ Ἀλέξανδρός φησιν, "ὅτι ὑποτίθεται μὲν ὁ γεωμέτρης τὸ τὸν κύκλον τῆς εὐθείας κατὰ σημεῖον ἅπτεσθαι ὡς ἀρχήν, ὁ δὲ Ἀντιφῶν ἀναιρεῖ τοῦτο." οὐ γὰρ ὑποτίθεται ὁ γεωμέτρης τοῦτο, ἀλλ' ἀποδείκνυσιν αὐτὸ ἐν τῷ τρίτῳ βιβλίῳ. ἄμεινον οὖν

[1] πάντα . . . εἰς: a lacuna in the text is satisfactorily filled, as Schenkl notes, if these words are supplied from Simplicius.

---

[a] Accounts differ about Antiphon's procedure, but it makes no difference to the result, which is to get a regular polygon approaching the circle as its limit. Themistius was

Examples are given by two men who tried to square the circle, Hippocrates of Chios and Antiphon. The attempt of Hippocrates is to be refuted. For, while preserving the principles, he commits a paralogism by squaring only that lune which is described about the side of the square inscribed in the circle, though including every lune that can be squared in the proof. But the geometer could have nothing to say against Antiphon, who inscribed an equilateral triangle in the circle,[a] and on each of the sides set up another triangle, an isosceles triangle with its vertex on the circumference of the circle, and continued this process, thinking that at some time he would make the side of the last triangle, although a straight line, coincide with the circumference.

Simplicius, *Commentary on Aristotle's Physics* A 2 (185 a 14), ed. Diels 54. 20–55. 24

Antiphon described a circle and inscribed some one of the (regular) polygons that can be inscribed therein. Suppose, for example, that the inscribed polygon is a square. . . . It is clear that the breach with the principles of geometry comes about not, as Alexander says, " because the geometer lays down as a hypothesis that a circle touches a straight line in one point [only], while Antiphon violates this." For the geometer does not lay this down as a hypothesis, but it is proved in the third book of the *Elements*.[b] It

the earliest of the commentators, and Heath considers his account " the authentic version." Philoponus makes Antiphon begin by inscribing a square, then an octagon and so on. Simplicius, as will be seen below, allows him to begin with any one of the regular polygons, but starts with the square as an example.

[b] Eucl. *Elem.* iii. 16.

λέγειν ἀρχὴν εἶναι τὸ ἀδύνατον εἶναι εὐθεῖαν ἐφαρμόσαι περιφερείᾳ, ἀλλ' ἡ μὲν ἐκτὸς κατὰ ἓν σημεῖον ἐφάψεται τοῦ κύκλου, ἡ δὲ ἐντὸς κατὰ δύο μόνον καὶ οὐ πλείω, καὶ ἡ ἐπαφὴ κατὰ σημεῖον γίνεται. καὶ μέντοι τέμνων ἀεὶ τὸ μεταξὺ τῆς εὐθείας καὶ τῆς τοῦ κύκλου περιφερείας ἐπίπεδον οὐ δαπανήσει αὐτὸ οὐδὲ καταλήψεταί ποτε τὴν τοῦ κύκλου περιφέρειαν, εἴπερ ἐπ' ἄπειρόν ἐστι διαιρετὸν τὸ ἐπίπεδον. εἰ δὲ καταλαμβάνει, ἀνῄρηταί τις ἀρχὴ γεωμετρικὴ ἡ λέγουσα ἐπ' ἄπειρον εἶναι τὰ μεγέθη διαιρετά. καὶ ταύτην καὶ ὁ Εὔδημος τὴν ἀρχὴν ἀναιρεῖσθαί φησιν ὑπὸ τοῦ Ἀντιφῶντος.

(ii.) *Bryson*

Alex. Aphr. *in Soph. El.* 11 (Aristot. 171 b 7), ed. Wallies 90. 10-21

Ἀλλ' ὁ τοῦ Βρύσωνος τετραγωνισμὸς τοῦ κύκλου ἐριστικός ἐστι καὶ σοφιστικός, ὅτι οὐκ ἐκ τῶν οἰκείων ἀρχῶν τῆς γεωμετρίας ἀλλ' ἔκ τινων κοινοτέρων. τὸ γὰρ περιγράφειν ἐκτὸς τοῦ κύκλου

[a] Heath (*H.G.M.* i. 222-223) comments: "The objection to Antiphon's statement is really no more than verbal: Euclid uses exactly the same construction in xii. 2, only he expresses the conclusion in a different way, saying that, if the process be continued far enough, the small segments left over will be together less than any assigned area. Antiphon in effect said the same thing, which again we express by saying that the circle is the *limit* of such an inscribed polygon when the number of its sides is indefinitely increased. Antiphon therefore deserves an honourable place in the history of geometry as having originated the idea of *exhausting* an area by means of inscribed regular polygons

would be better therefore to say that the principle is that a straight line cannot coincide with the circumference, a straight line drawn from outside the circle touching it in one point only, a straight line drawn from inside cutting it in two points and not more, and tangential contact being in one point only. Now continual division of the space between the straight line and the circumference of the circle will never exhaust it nor ever reach the circumference of the circle, if the space is really divisible without limit. For if the circumference could be reached, the geometrical principle that magnitudes are divisible without limit would be violated. This was the principle which Eudemus says was violated by Antiphon.[a]

(ii.) *Bryson*[b]

Alexander, *Commentary on Aristotle's Sophistic Refutations* 11 (171 b 7), ed. Wallies 90. 10-21

But Bryson's quadrature of the circle is eristic and sophistical, because he proceeds not from principles peculiar to geometry but from wider principles. For to circumscribe a square about the circle and to

with an ever-increasing number of sides, an idea upon which Eudoxus founded his epoch-making *method of exhaustion*. The practical value of Antiphon's construction is illustrated by Archimedes' treatise on the *Measurement of a Circle* [reproduced below] . . . The same construction starting from a square was likewise the basis of Vieta's expression for $\frac{2}{\pi}$, namely,

$$\frac{2}{\pi}=\cos\frac{\pi}{4}\cdot\cos\frac{\pi}{8}\cdot\cos\frac{\pi}{16}\cdots$$
$$=\sqrt{\tfrac{1}{2}}\cdot\sqrt{\tfrac{1}{2}\left(1+\sqrt{\tfrac{1}{2}}\right)}\cdot\sqrt{\tfrac{1}{2}\left(1+\sqrt{\tfrac{1}{2}\left(1+\sqrt{\tfrac{1}{2}}\right)}\right)}\ldots\text{"}$$

[b] Bryson was a pupil either of Socrates or of Euclid of Megara.

τετράγωνον καὶ ἐντὸς ἐγγράφειν ἕτερον καὶ μεταξὺ τῶν δύο τετραγώνων ἕτερον τετράγωνον, εἶτα λέγειν ὅτι ὁ μεταξὺ τῶν δύο τετραγώνων κύκλος, ὁμοίως δὲ καὶ τὸ μεταξὺ τῶν δύο τετραγώνων τετράγωνον τοῦ μὲν ἐκτὸς τετραγώνου ἐλάττονά εἰσι τοῦ δὲ ἐντὸς μείζονα, τὰ δὲ τῶν αὐτῶν μείζονα καὶ ἐλάττονα ἴσα ἐστίν, ἴσος ἄρα ὁ κύκλος καὶ τὸ τετράγωνον, ἔκ τινων κοινῶν ἀλλὰ καὶ ψευδῶν ἐστι, κοινῶν μέν, ὅτι καὶ ἐπ' ἀριθμῶν καὶ χρόνων καὶ τόπων καὶ ἄλλων κοινῶν ἁρμόσοι ἄν, ψευδῶν δέ, ὅτι ὀκτὼ καὶ ἐννέα τῶν δέκα καὶ ἑπτὰ ἐλάττονες καὶ μείζονές εἰσι καὶ ὅμως οὐκ εἰσὶν ἴσοι.

(iii.) *Archimedes*

Procl. *in Eucl.* i., ed. Kroll 422. 24–423. 5

Ἐκ τούτου δὲ οἶμαι τοῦ προβλήματος ἐπαχθέντες οἱ παλαιοὶ καὶ τὸν τοῦ κύκλου τετραγωνισμὸν ἐζήτησαν. εἰ γὰρ παραλληλόγραμμον ἴσον εὑρίσκεται παντὶ εὐθυγράμμῳ, ζητήσεως ἄξιον, μὴ καὶ τὰ εὐθύγραμμα τοῖς περιφερογράμμοις ἴσα δεικνύναι δυνατόν. καὶ ὁ Ἀρχιμήδης ἔδειξεν, ὅτι πᾶς κύκλος ἴσος ἐστὶ τριγώνῳ ὀρθογωνίῳ, οὗ ἡ μὲν ἐκ κέντρου ἴση ἐστὶν μιᾷ τῶν περὶ τὴν ὀρθήν, ἡ δὲ περίμετρος τῇ βάσει.

Archim. *Dim. Circ.*, Archim. ed. Heiberg i. 232-242

α'

Πᾶς κύκλος ἴσος ἐστὶ τριγώνῳ ὀρθογωνίῳ, οὗ

[a] Bryson marks a step beyond Antiphon because he conceived the circle as intermediate in area between an inscribed and an escribed polygon, an idea which was powerfully developed by Archimedes. The manner in which he took a square intermediate between the inscribed and escribed

inscribe another[a] within and between the two squares to take another square, and then to say that the circle is intermediate between the two squares, and similarly that the square between the two squares is less than the outside square but greater than the inside and that, since things which are greater and less than the same things are equal, therefore the circle and the square are equal, is to proceed from wider principles (than those of geometry) and false ones; wider, because the argument would apply to numbers and times and spaces and other entities, false, because eight and nine are respectively less and greater than ten and seven and nevertheless are not equal.

(iii.) *Archimedes*

Proclus, *On Euclid* i., ed. Kroll 422. 24–423. 5

I think it was in consequence of this problem[b] that the ancient geometers were led to investigate the squaring of the circle. For if a parallelogram is found equal to any rectilineal figure, it is worth inquiring whether it be not also possible to prove rectilineal figures equal to circular. Archimedes in fact proved that any circle is equal to a right-angled triangle wherein one of the sides about the right-angle is equal to the radius and the base to the perimeter.

Archimedes, *Measurement of a Circle*, Archim. ed. Heiberg i. 232-242

Prop. 1

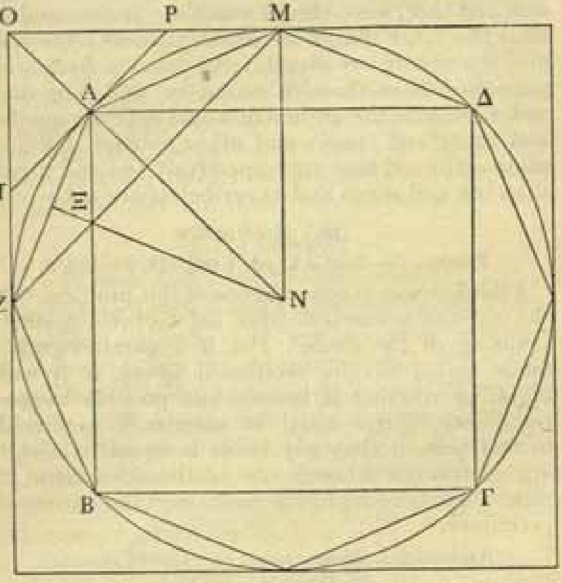
Any circle is equal to a right-angled triangle in which

squares is unknown. Some have assumed that it was the arithmetic mean, others the geometric (see Heath, *H.G.M.* i. 223, 224).

[b] Eucl. i. 45. "To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure."

ἡ μὲν ἐκ τοῦ κέντρου ἴση μιᾷ τῶν περὶ τὴν ὀρθήν, ἡ δὲ περίμετρος τῇ βάσει.

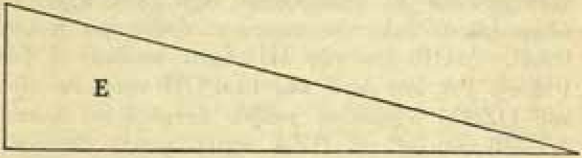
Ἐχέτω ὁ ΑΒΓΔ κύκλος τριγώνῳ τῷ Ε, ὡς ὑπόκειται· λέγω, ὅτι ἴσος ἐστίν.



Εἰ γὰρ δυνατόν, ἔστω μείζων ὁ κύκλος, καὶ ἐγγεγράφθω τὸ ΑΓ τετράγωνον, καὶ τετμήσθωσαν αἱ περιφέρειαι δίχα, καὶ ἔστω τὰ τμήματα ἤδη ἐλάσσονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ κύκλος τοῦ τριγώνου· τὸ εὐθύγραμμον ἄρα ἔτι τοῦ τριγώνου ἐστὶ μεῖζον. εἰλήφθω κέντρον τὸ Ν καὶ κάθετος

one of the sides about the right angle is equal to the radius, and the base is equal to the circumference.

Let the circle ΑΒΓΔ have to the triangle E the stated relation ; I say that it is equal.



For, if possible, let the circle be greater, and let the square ΑΓ be inscribed, and let the arcs be divided into equal parts [and let ΒΖ, ΖΑ, ΑΜ, ΜΔ, etc., be drawn],[a] and let the segments be less than the excess by which the circle exceeds the triangle.[b] The rectilineal figure is therefore greater than the triangle.

[a] Heiberg's note is: "Tale aliquid Archimedes sine dubio addiderat: Omnino in toto hoc opusculo genus dicendi et exponendi brevitate tam negligenti laborat, ut manum excerptoris potius quam Archimedis agnoscas."

[b] That this can be done is shown in Eucl. *Elem.* xii. 2, depending on x. 1. The latter theorem was probably discovered by Eudoxus, but is commonly known as the "Axiom of Archimedes" from his repeated use of it.

ἡ ΝΞ· ἐλάσσων ἄρα ἡ ΝΞ τῆς τοῦ τριγώνου πλευρᾶς. ἔστιν δὲ καὶ ἡ περίμετρος τοῦ εὐθυγράμμου τῆς λοιπῆς ἐλάττων, ἐπεὶ καὶ τῆς τοῦ κύκλου περιμέτρου· ἔλαττον ἄρα τὸ εὐθύγραμμον τοῦ Ε τριγώνου· ὅπερ ἄτοπον.

Ἔστω δὲ ὁ κύκλος, εἰ δυνατόν, ἐλάσσων τοῦ Ε τριγώνου, καὶ περιγεγράφθω τὸ τετράγωνον, καὶ τετμήσθωσαν αἱ περιφέρειαι δίχα, καὶ ἤχθωσαν ἐφαπτόμεναι διὰ τῶν σημείων· ὀρθὴ ἄρα ἡ ὑπὸ ΟΑΡ. ἡ ΟΡ ἄρα τῆς ΜΡ ἐστιν μείζων· ἡ γὰρ ΡΜ τῇ ΡΑ ἴση ἐστί· καὶ τὸ ΡΟΠ τρίγωνον ἄρα τοῦ ΟΖΑΜ σχήματος μεῖζόν ἐστιν ἢ τὸ ἥμισυ. λελείφθωσαν οἱ τῷ ΠΖΑ τομεῖ ὅμοιοι ἐλάσσους τῆς ὑπεροχῆς, ᾗ ὑπερέχει τὸ Ε τοῦ ΑΒΓΔ κύκλου· ἔτι ἄρα τὸ περιγεγραμμένον εὐθύγραμμον τοῦ Ε ἐστιν ἔλασσον· ὅπερ ἄτοπον· ἔστιν γὰρ μεῖζον, ὅτι ἡ μὲν ΝΑ ἴση ἐστὶ τῇ καθέτῳ τοῦ τριγώνου, ἡ δὲ περίμετρος μείζων ἐστὶ τῆς βάσεως τοῦ τριγώνου. ἴσος ἄρα ὁ κύκλος τῷ Ε τριγώνῳ.

γ′

Παντὸς κύκλου ἡ περίμετρος τῆς διαμέτρου τριπλασίων ἐστὶ καὶ ἔτι ὑπερέχει ἐλάσσονι μὲν ἢ ἑβδόμῳ μέρει τῆς διαμέτρου, μείζονι δὲ ἢ δέκα ἑβδομηκοστομόνοις.

---

[a] *i.e.*, the space between the arc ΖΑ of the circle and the sides ΖΠ, ΠΑ of the escribed polygon. The name given to this figure, τομεύς, is more properly used of a *sector* of a circle, and Heiberg notes: "τομεῖ Archimedes non scripsit pro ἀποτμήματι." The process, it is not quite clearly stated

Let N be the centre, and NΞ perpendicular [to ZA]; NΞ is then less than the side of the triangle. But the perimeter of the rectilineal figure is also less than the other side, since it is less than the perimeter of the circle. The rectilineal figure is therefore less than the triangle E; which is absurd.

Let the circle be, if possible, less than the triangle E, and let the square be circumscribed, and let the arcs be divided into equal parts, and through the points [of division] let tangents be drawn; the angle OAP is therefore right. Therefore OP is greater than MP; for PM is equal to PA; and the triangle POΠ is greater than half the figure OZAM. Let the spaces left between the circle and the circumscribed polygon, such as the figure[a] ΠZA, be less than the excess by which E exceeds the circle ΑΒΓΔ. Therefore the circumscribed rectilineal figure is now less than E; which is absurd; for it is greater, because NA is equal to the perpendicular of the triangle, while the perimeter is greater than the base of the triangle. The circle is therefore equal to the triangle E.

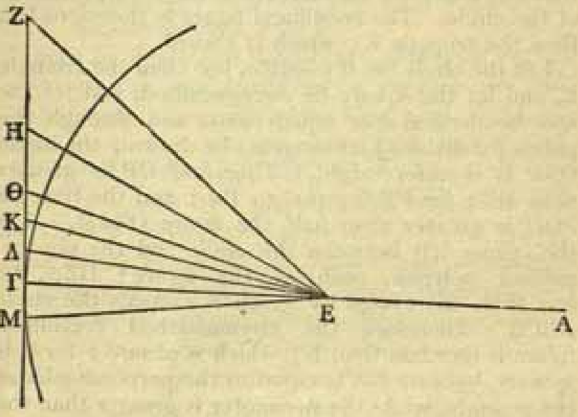
Prop. *3*[b]

The circumference of any circle is greater than three times the diameter and exceeds it by a quantity less than the seventh part of the diameter but greater than ten seventy-first parts.

in the Greek, is to be *continued until* the escribed polygon is such that the spaces left between it and the circle are less than the excess of E over the circle. That this can be done follows from the "Axiom of Archimedes," Eucl. *Elem.* x. 1.

[b] The order of the propositions in the manuscripts is manifestly wrong. Props. 2 and 3 must be interchanged.

[a] Ἔστω κύκλος καὶ διάμετρος ἡ ΑΓ καὶ κέντρον τὸ Ε καὶ ἡ ΓΛΖ ἐφαπτομένη καὶ ἡ ὑπὸ ΖΕΓ τρίτου ὀρθῆς· ἡ ΕΖ ἄρα πρὸς ΖΓ λόγον ἔχει, ὃν



$\overline{\tau\varsigma}$ πρὸς $\overline{\rho\nu\gamma}$, ἡ δὲ ΕΓ πρὸς [τὴν] ΓΖ λόγον ἔχει, ὃν $\overline{\sigma\xi\epsilon}$ πρὸς $\overline{\rho\nu\gamma}$. τετμήσθω οὖν ἡ ὑπὸ ΖΕΓ δίχα τῇ ΕΗ· ἔστιν ἄρα, ὡς ἡ ΖΕ πρὸς ΕΓ, ἡ ΖΗ πρὸς ΗΓ [καὶ ἐναλλὰξ καὶ συνθέντι]. ὡς ἄρα συναμφότερος ἡ ΖΕ, ΕΓ πρὸς ΖΓ, ἡ ΕΓ πρὸς ΓΗ· ὥστε ἡ ΓΕ πρὸς ΓΗ μείζονα λόγον ἔχει ἤπερ $\overline{\phi o \alpha}$ πρὸς $\overline{\rho\nu\gamma}$. ἡ ΕΗ ἄρα πρὸς ΗΓ δυνάμει λόγον ἔχει, ὃν $\overset{\lambda\delta}{\mathrm{M}}\ \overline{,\theta\upsilon\nu}$ πρὸς $\overset{\beta}{\mathrm{M}}\ \overline{,\gamma\upsilon\theta}$· μήκει ἄρα, ὃν

---

[a] As Eutocius explains in his commentary on this passage (Archim. ed. Heiberg iii. 234), if ΕΖ is represented by 306 and ΓΖ by 153, then by Pythagoras's theorem $ΕΓ^2 = 306^2 - 153^2 = 70227$. Since $265^2 = 70225$, ΕΓ is therefore 265

Let there be a circle with diameter ΑΓ and centre Ε, and let ΓΑΖ be a tangent and the angle ΖΕΓ one-third of a right angle. Then

$$\mathrm{E\Gamma : \Gamma Z}\ [= \sqrt{3} : 1] > 265 : 153^{a} \quad . \quad . \quad (1)$$

and

$$\mathrm{EZ : Z\Gamma}\ [= 2 : 1] = 306 : 153 \quad . \quad . \quad (2)$$

Now let ∠ΖΕΓ be bisected by ΕΗ. It follows that

$$\mathrm{ZE : E\Gamma} = \mathrm{ZH : H\Gamma}\ [\text{Eucl. vi. 3}$$

so that

$$\begin{aligned} [\mathrm{ZE + E\Gamma : E\Gamma} &= \mathrm{ZH + H\Gamma : H\Gamma} \\ &= \mathrm{Z\Gamma : H\Gamma}, \text{ or}] \\ \mathrm{ZE + E\Gamma : Z\Gamma} &= \mathrm{E\Gamma : H\Gamma}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{\Gamma E : \Gamma H}\ & [= \mathrm{E\Gamma + ZE : Z\Gamma} \\ & > 265 + 306 : 153, \text{ by (1) and (2)}] \\ & > 571 : 153 \quad . \quad . \quad (3) \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{EH^2 : H\Gamma^2}\ & [= \mathrm{E\Gamma^2 + \Gamma H^2 : H\Gamma^2} \\ & > 571^2 + 153^2 : 153^2] \\ & > 349450 : 23409, \end{aligned}$$

and a "minute and imperceptible fraction" (*μόριον ἐλάχιστον καὶ ἀνεπαίσθητον*). As the sides of the triangle are in the ratio 1, $\sqrt{3}$, 2, this is equivalent to saying that $\sqrt{3} > \frac{265}{153}$. In the second part of the proof Archimedes assumes that $\sqrt{3} < \frac{1351}{780}$. The way in which he makes these assumptions, without explanation of any kind, shows that they were common in his day, and much ingenuity has been spent in devising processes by which they may have been reached. *v.* Heath, *The Works of Archimedes*, lxxx-lxxxiv, xc-xcix.

Eutocius fully explains the arithmetical working, where Archimedes merely sets down the results. In the translation the necessary working, where not given by Archimedes, is shown in square brackets. In the Greek text as we have it a few equalities are given where the argument requires inequalities. The translation reproduces what Archimedes must have written.

φϛα η′ πρὸς ρνγ. πάλιν δίχα ἡ ὑπὸ ΗΕΓ τῇ ΕΘ· διὰ τὰ αὐτὰ ἄρα ἡ ΕΓ πρὸς ΓΘ μείζονα λόγον ἔχει ἢ ὃν ͵αρξβ η′ πρὸς ρνγ· ἡ ΘΕ ἄρα πρὸς ΘΓ μείζονα λόγον ἔχει ἢ ὃν ͵αροβ η′ πρὸς ρνγ. ἔτι δίχα ἡ ὑπὸ ΘΕΓ τῇ ΕΚ· ἡ ΕΓ ἄρα πρὸς ΓΚ μείζονα λόγον ἔχει ἢ ὃν ͵βτλδ δ′ πρὸς ρνγ· ἡ ΕΚ ἄρα πρὸς

so that $\quad EH : H\Gamma \quad > 591\tfrac{1}{8} : 153 \quad . \quad . \quad (4)$

Again, let $\angle HE\Gamma$ be bisected by $E\Theta$; then by the same reasoning

$\quad$ [$HE : E\Gamma \quad = H\Theta : \Theta\Gamma$ [Eucl. vi. 3

so that $\quad HE + E\Gamma : E\Gamma \quad = H\Theta + \Theta\Gamma : \Theta\Gamma$

$\quad = H\Gamma : \Gamma\Theta,$

or $\quad HE + E\Gamma : H\Gamma \quad = E\Gamma : \Gamma\Theta.$

Therefore] $E\Gamma : \Gamma\Theta \quad$ [$= \Gamma E + EH : H\Gamma$

$\quad > 571 + 591\tfrac{1}{8} : 153,$

by (3) and (4),]

$\quad > 1162\tfrac{1}{8} : 153 \quad . \quad (5)$

[Hence $\quad \Theta E^2 : \Gamma\Theta^2 \quad = E\Gamma^2 + \Gamma\Theta^2 : \Gamma\Theta^2$

$\quad > 1162\tfrac{1}{8}^2 + 153^2 : 153^2$

$\quad > 1350534\tfrac{33}{64} + 23409 : 23409$

$\quad > 1373943\tfrac{33}{64} : 23409,$]

so that $\quad \Theta E : \Theta\Gamma \quad > 1172\tfrac{1}{8} : 153 \quad . \quad (6)$

Again, let $\Theta E\Gamma$ be bisected by EK.

Then $\quad$ [$\Theta E : E\Gamma \quad = \Theta K : K\Gamma \quad .$ [Eucl. vi. 3

so that $\quad \Theta E + E\Gamma : E\Gamma \quad = \Theta K + K\Gamma : K\Gamma$

$\quad = \Theta\Gamma : \Gamma K,$ or]

$E\Gamma : \Gamma K \quad$ [$= E\Gamma + \Theta E : \Theta\Gamma$

$\quad > 1162\tfrac{1}{8} + 1172\tfrac{1}{8} : 153,$

by (5) and (6),]

$\quad > 2334\tfrac{1}{4} : 153 \quad . \quad . \quad (7)$

[Hence $\quad EK^2 : \Gamma K^2 \quad = E\Gamma^2 + \Gamma K^2 : \Gamma K^2$

$\quad > 2334\tfrac{1}{4}^2 + 153^2 : 153^2$

$\quad > 5472132\tfrac{1}{16} : 23409,$]

ΓΚ μείζονα ἢ ὃν ͵βτλθ δʹ πρὸς ρνγ. ἔτι δίχα ἡ ὑπὸ ΚΕΓ τῇ ΛΕ· ἡ ΕΓ ἄρα πρὸς ΛΓ μείζονα [μήκει] λόγον ἔχει ἤπερ τὰ ͵δχογ ∠ʹ πρὸς ρνγ. ἐπεὶ οὖν ἡ ὑπὸ ΖΕΓ τρίτου οὖσα ὀρθῆς τέτμηται τετράκις δίχα, ἡ ὑπὸ ΛΕΓ ὀρθῆς ἐστι μηʹ. κείσθω οὖν αὐτῇ ἴση πρὸς τῷ Ε ἡ ὑπὸ ΓΕΜ· ἡ ἄρα ὑπὸ ΛΕΜ ὀρθῆς ἐστι κδʹ· καὶ ἡ ΛΜ ἄρα εὐθεῖα τοῦ περὶ τὸν κύκλον ἐστὶ πολυγώνου πλευρὰ πλευρὰς ἔχοντος ϙϛ. ἐπεὶ οὖν ἡ ΕΓ πρὸς τὴν ΓΛ ἐδείχθη μείζονα λόγον ἔχουσα ἤπερ ͵δχογ ∠ʹ πρὸς ρνγ, ἀλλὰ τῆς μὲν ΕΓ διπλῆ ἡ ΑΓ, τῆς δὲ ΓΛ διπλασίων ἡ ΛΜ, καὶ ἡ ΑΓ ἄρα πρὸς τὴν τοῦ ϙϛ-γώνου περίμετρον μείζονα λόγον ἔχει ἤπερ ͵δχογ ∠ʹ πρὸς $\overset{\alpha}{\mathrm{M}}$ ͵δχπη. καί ἐστιν τριπλάσια, καὶ ὑπερέχουσιν χξζ ∠ʹ, ἅπερ τῶν ͵δχογ ∠ʹ ἐλάττονά ἐστιν ἢ τὸ ἕβδομον· ὥστε τὸ πολύγωνον τὸ περὶ τὸν κύκλον τῆς διαμέτρου ἐστὶ τριπλάσιον καὶ ἐλάττονι ἢ τῷ ἑβδόμῳ μέρει μεῖζον· ἡ τοῦ κύκλου ἄρα περίμετρος πολὺ μᾶλλον ἐλάσσων ἐστὶν ἢ τριπλασίων καὶ ἑβδόμῳ μέρει μείζων.

Ἔστω κύκλος καὶ διάμετρος ἡ ΑΓ, ἡ δὲ ὑπὸ ΒΑΓ τρίτου ὀρθῆς· ἡ ΑΒ ἄρα πρὸς ΒΓ ἐλάσσονα λόγον ἔχει ἢ ὃν ͵ατνα πρὸς ψπ [ἡ δὲ ΑΓ πρὸς ΓΒ, ὃν ͵αφξ πρὸς ψπ]. δίχα ἡ ὑπὸ ΒΑΓ τῇ ΑΗ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ ΒΑΗ τῇ ὑπὸ ΗΓΒ, ἀλλὰ

so that	$EK : \Gamma K$	$> 2339\frac{1}{4} : 153$	. . (8)

Again, let $\angle KE\Gamma$ be bisected by ΛE.

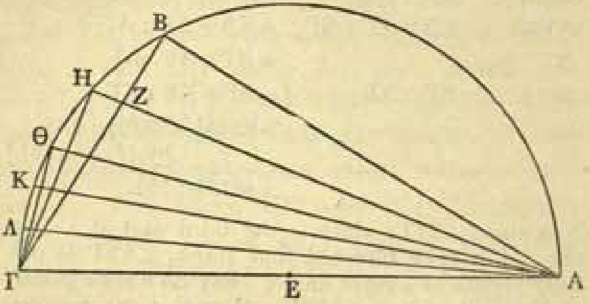
Then	[$KE : E\Gamma$	$= K\Lambda : \Lambda\Gamma$	[Eucl. vi. 3
so that	$KE + E\Gamma : E\Gamma$	$= K\Lambda + \Lambda\Gamma : \Lambda\Gamma$	
		$= K\Gamma : \Lambda\Gamma$, or]	
	$E\Gamma : \Lambda\Gamma$	[$= E\Gamma + KE : K\Gamma$	
		$> 2334\frac{1}{4} + 2339\frac{1}{4} : 153$, by (7) and (8),]	
		$> 4673\frac{1}{2} : 153$.	

Now since $\angle ZE\Gamma$, which is the third part of a right angle, has been bisected four times, $\angle \Lambda E\Gamma$ is one forty-eighth of a right angle. Let $\angle \Gamma EM$ be placed at E equal to it. $\angle \Lambda EM$ is therefore one twenty-fourth of a right angle. And ΛM is therefore the side of a polygon escribed to the circle and having ninety-six sides. Since $E\Gamma : \Gamma\Lambda$ was proved to be greater than $4673\frac{1}{2} : 153$ and $A\Gamma = 2E\Gamma$, $\Lambda M = 2\Gamma\Lambda$, the ratio of $A\Gamma$ to the perimeter of the 96-sided polygon is greater than [$4673\frac{1}{2} : 96 . 153$, or] $4673\frac{1}{2} : 14688$. And the ratio [$14688 : 4673\frac{1}{2}$] is greater than 3, being in excess by $667\frac{1}{2}$, which is less than the seventh part of $4673\frac{1}{2}$; so that the [perimeter of the] escribed polygon is greater than three times the diameter by less than the seventh part; *a fortiori* therefore the circumference of the circle is less than $3\frac{1}{7}$ times the diameter.

Let there be a circle with diameter $A\Gamma$ and $\angle BA\Gamma$ one-third of a right angle. Then $AB : B\Gamma$ [$= \sqrt{3} : 1$] $< 1351 : 780$.[a] Let $BA\Gamma$ be bisected by AH. Now since $\angle BAH = \angle H\Gamma B$ and $\angle BAH = \angle HA\Gamma$, there-

[a] See *supra*, p. 322 n. *a*.

καὶ τῇ ὑπὸ ΗΑΓ, καὶ ἡ ὑπὸ ΗΓΒ τῇ ὑπὸ ΗΑΓ ἐστιν ἴση. καὶ κοινὴ ἡ ὑπὸ ΑΗΓ ὀρθή· καὶ τρίτη



ἄρα ἡ ὑπὸ ΗΖΓ τρίτῃ τῇ ὑπὸ ΑΓΗ ἴση. ἰσογώνιον ἄρα τὸ ΑΗΓ τῷ ΓΗΖ τριγώνῳ· ἔστιν ἄρα, ὡς ἡ ΑΗ πρὸς ΗΓ, ἡ ΓΗ πρὸς ΗΖ καὶ ἡ ΑΓ πρὸς ΓΖ. ἀλλ' ὡς ἡ ΑΓ πρὸς ΓΖ, [καὶ] συναμφότερος ἡ ΓΑΒ πρὸς ΒΓ· καὶ ὡς συναμφότερος ἄρα ἡ ΒΑΓ πρὸς ΒΓ, ἡ ΑΗ πρὸς ΗΓ. διὰ τοῦτο οὖν ἡ ΑΗ πρὸς [τὴν] ΗΓ ἐλάσσονα λόγον ἔχει ἤπερ ͵βϡια πρὸς ψπ, ἡ δὲ ΑΓ πρὸς τὴν ΓΗ ἐλάσσονα ἢ ὃν ͵γιγ ∠′ δ′ πρὸς ψπ. δίχα ἡ ὑπὸ ΓΑΗ τῇ ΑΘ· ἡ ΑΘ ἄρα διὰ τὰ αὐτὰ πρὸς τὴν ΘΓ ἐλάσσονα λόγον ἔχει ἢ ὃν ͵εϡκδ ∠′ δ′ πρὸς ψπ ἢ ὃν ͵αωκγ πρὸς σμ· ἑκατέρα γὰρ ἑκατέρας δ̅ ιγ′· ὥστε ἡ ΑΓ πρὸς τὴν ΓΘ ἢ ὃν ͵αωλη θ̅ ια′ πρὸς

fore $\angle \mathrm{H\Gamma B} = \angle \mathrm{HA\Gamma}$. And the right angle $\mathrm{AH\Gamma}$ is common. Therefore the third angle $\mathrm{HZ\Gamma}$ is equal to the third angle $\mathrm{A\Gamma H}$. The triangle $\mathrm{AH\Gamma}$ is therefore equiangular with the triangle $\mathrm{\Gamma HZ}$; therefore

$$\mathrm{AH} : \mathrm{H\Gamma} = \mathrm{\Gamma H} : \mathrm{HZ} = \mathrm{A\Gamma} : \mathrm{\Gamma Z}.$$

But $\quad \mathrm{A\Gamma} : \mathrm{\Gamma Z} = \mathrm{\Gamma A} + \mathrm{AB} : \mathrm{B\Gamma}$.

Therefore $\quad \mathrm{BA} + \mathrm{A\Gamma} : \mathrm{B\Gamma} = \mathrm{AH} : \mathrm{H\Gamma}$.

[But $\quad \mathrm{BA} : \mathrm{B\Gamma} < 1351 : 780$, as stated above,

while $\quad \mathrm{A\Gamma} : \mathrm{B\Gamma} = 2 : 1$

$\quad = 1560 : 780$.]

Therefore $\quad \mathrm{AH} : \mathrm{H\Gamma}$ $[= 1351 + 1560 : 780]$

$\quad < 2911 : 780 \quad . \quad . \quad . \quad . \quad (1a)$

[Hence $\quad \mathrm{A\Gamma}^2 : \mathrm{\Gamma H}^2 = \mathrm{AH}^2 + \mathrm{H\Gamma}^2 : \mathrm{\Gamma H}^2$

$\quad < 2911^2 + 780^2 : 780^2$

$\quad < 9082321 : 608400$,]

so that $\quad \mathrm{A\Gamma} : \mathrm{\Gamma H} < 3013\tfrac{3}{4} : 780 \quad . \quad . \quad . \quad . \quad (2a)$

Let $\angle \mathrm{\Gamma AH}$ be bisected by $\mathrm{A\Theta}$. By the same reasoning

$\mathrm{A\Theta} : \mathrm{\Theta\Gamma}$ $[= \mathrm{A\Gamma} + \mathrm{AH} : \mathrm{\Gamma H}$

$\quad < 3013\tfrac{3}{4} + 2911 : 780$, by $(1a)$ and $(2a)$,]

$\quad < 5924\tfrac{3}{4} : 780$

$\quad < \tfrac{4}{13} \cdot 5924\tfrac{3}{4} : \tfrac{4}{13} \cdot 780$

$\quad < 1823 : 240 \quad . \quad . \quad . \quad . \quad (3a)$

[Hence $\quad \mathrm{A\Gamma}^2 : \mathrm{\Gamma\Theta}^2 = \mathrm{A\Theta}^2 + \mathrm{\Gamma\Theta}^2 : \mathrm{\Gamma\Theta}^2$

$\quad < 1823^2 + 240^2 : 240^2$

$\quad < 3380929 : 57600$.]

Therefore $\quad \mathrm{A\Gamma} : \mathrm{\Gamma\Theta} < 1838\tfrac{9}{11} : 240 \quad . \quad . \quad . \quad (4a)$

$\overline{\sigma\mu}$. ἔτι δίχα ἡ ὑπὸ ΘΑΓ τῇ ΚΑ· καὶ ἡ ΑΚ πρὸς τὴν ΚΓ ἐλάσσονα [ἄρα] λόγον ἔχει ἢ ὃν $\overline{,\alpha\zeta}$ πρὸς $\overline{\xi\varsigma}$· ἑκατέρα γὰρ ἑκατέρας $\overline{\iota\alpha}$ μ'· ἡ ΑΓ ἄρα πρὸς [τὴν] ΚΓ ἢ ὃν $\overline{,\alpha\theta}$ ς' πρὸς $\overline{\xi\varsigma}$. ἔτι δίχα ἡ ὑπὸ ΚΑΓ τῇ ΛΑ· ἡ ΑΛ ἄρα πρὸς [τὴν] ΛΓ ἐλάσσονα λόγον ἔχει ἢ ὃν τὰ $\overline{,\beta\iota\varsigma}$ ς' πρὸς $\overline{\xi\varsigma}$, ἡ δὲ ΑΓ πρὸς ΓΛ ἐλάσσονα ἢ τὰ $\overline{,\beta\iota\zeta}$ δ' πρὸς $\overline{\xi\varsigma}$. ἀνάπαλιν ἄρα ἡ περίμετρος τοῦ πολυγώνου πρὸς τὴν διάμετρον μείζονα λόγον ἔχει ἤπερ $\overline{,\varsigma\tau\lambda\varsigma}$ πρὸς $\overline{,\beta\iota\zeta}$ δ', ἅπερ τῶν $\overline{,\beta\iota\zeta}$ δ' μείζονά ἐστιν ἢ τριπλασίονα καὶ δέκα $\overline{\text{οα}}'$· καὶ ἡ περίμετρος ἄρα τοῦ $\overline{\text{϶ς}}$-γώνου τοῦ ἐν τῷ κύκλῳ τῆς διαμέτρου τριπλασίων ἐστὶ καὶ μείζων ἢ $\bar{\iota}$ $\text{οα}'$· ὥστε καὶ ὁ κύκλος ἔτι μᾶλλον τριπλασίων ἐστὶ καὶ μείζων ἢ $\bar{\iota}$ $\text{οα}'$.

Ἡ ἄρα τοῦ κύκλου περίμετρος τῆς διαμέτρου

Further, let $\angle \Theta \mathrm{A}\Gamma$ be bisected by KA.

Then $\quad \mathrm{AK} : \mathrm{K}\Gamma \quad [= \mathrm{A}\Gamma + \mathrm{A}\Theta : \Gamma\Theta$

$$< 1838\tfrac{9}{11} + 1823 : 24, \text{ by } (3a) \text{ and } (4a),$$

$$< 3661\tfrac{9}{11} : 240]$$

$$< \tfrac{11}{40} \cdot 3661\tfrac{9}{11} : \tfrac{11}{40} \cdot 240$$

$$< 1007 : 66 \quad . \quad . \quad . \quad . \quad (5a)$$

[Hence $\quad \mathrm{A}\Gamma^2 : \mathrm{K}\Gamma^2 = \mathrm{AK}^2 + \mathrm{K}\Gamma^2 : \mathrm{K}\Gamma^2$

$$< 1007^2 + 66^2 : 66^2$$

$$< 1018405 : 4356.]$$

Therefore $\quad \mathrm{A}\Gamma : \mathrm{K}\Gamma \quad < 1009\tfrac{1}{6} : 66 \quad . \quad . \quad . \quad . \quad (6a)$

Further, let $\angle \mathrm{KA}\Gamma$ be bisected by ΛA.

Then $\quad \mathrm{A}\Lambda : \Lambda\Gamma \quad [= \Gamma\mathrm{A} + \mathrm{AK} : \Gamma\mathrm{K}$

$$< 1009\tfrac{1}{6} + 1007 : 66, \text{ by } (5a) \text{ and } (6a),]$$

$$< 2016\tfrac{1}{6} : 66.$$

[Hence $\quad \mathrm{A}\Gamma^2 : \Gamma\Lambda^2 = \mathrm{A}\Lambda^2 + \Lambda\Gamma^2 : \Gamma\Lambda^2$

$$< 2016\tfrac{1}{6}^2 + 66^2 : 66^2$$

$$< 4069284\tfrac{1}{36} : 4356.]$$

Therefore $\quad \mathrm{A}\Gamma : \Gamma\Lambda \quad < 2017\tfrac{1}{4} : 66,$

and *invertendo* [$\Gamma\Lambda : \mathrm{A}\Gamma > 66 : 2017\tfrac{1}{4}$.

But $\Gamma\Lambda$ is the side of a polygon of 96 sides; and accordingly] the perimeter of the polygon bears to the diameter a ratio greater than [$96 . 66 : 2017\tfrac{1}{4}$, or] $6336 : 2017\tfrac{1}{4}$, which is greater than $3\tfrac{10}{71}$. Therefore the perimeter of the 96-sided polygon is greater than $3\tfrac{10}{71}$ times the diameter, so that *a fortiori* the circle is greater than $3\tfrac{10}{71}$ times the diameter.

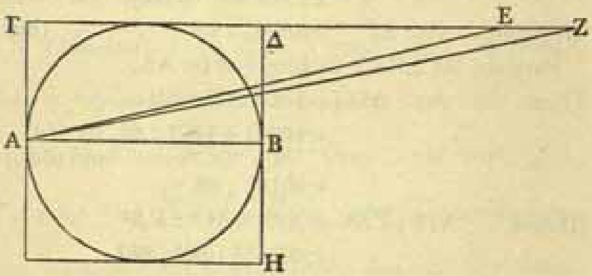
The perimeter of the circle is therefore more than

τριπλασίων ἐστὶ καὶ ἐλάσσονι μὲν ἢ ἑβδόμῳ μέρει, μείζονι δὲ ἢ $\overline{\iota}$ οα´ μείζων.

β´

Ὁ κύκλος πρὸς τὸ ἀπὸ τῆς διαμέτρου τετράγωνον λόγον ἔχει, ὃν $\overline{\iota\alpha}$ πρὸς $\overline{\iota\delta}$.

Ἔστω κύκλος, οὗ διάμετρος ἡ ΑΒ, καὶ περιγεγράφθω τετράγωνον τὸ ΓΗ, καὶ τῆς ΓΔ διπλῆ ἡ ΔΕ, ἕβδομον δὲ ἡ ΕΖ τῆς ΓΔ. ἐπεὶ οὖν τὸ



ΑΓΕ πρὸς τὸ ΑΓΔ λόγον ἔχει, ὃν $\overline{\kappa\alpha}$ πρὸς $\overline{\zeta}$, πρὸς δὲ τὸ ΑΕΖ τὸ ΑΓΔ λόγον ἔχει, ὃν ἑπτὰ πρὸς ἕν, τὸ ΑΓΖ πρὸς τὸ ΑΓΔ ἐστιν, ὡς $\overline{\kappa\beta}$ πρὸς $\overline{\zeta}$. ἀλλὰ τοῦ ΑΓΔ τετραπλάσιόν ἐστι τὸ ΓΗ τετράγωνον, τὸ δὲ ΑΓΔΖ τρίγωνον τῷ ΑΒ κύκλῳ ἴσον ἐστὶν [ἐπεὶ ἡ μὲν ΑΓ κάθετος ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου, ἡ δὲ βάσις τῆς διαμέτρου τριπλασίον καὶ τῷ ζ´ ἔγγιστα ὑπερέχουσα δειχθήσεται][1]· ὁ κύκλος οὖν πρὸς τὸ ΓΗ τετράγωνον λόγον ἔχει, ὃ $\overline{\iota\alpha}$ πρὸς $\overline{\iota\delta}$.

three times the diameter, exceeding by a quantity less than the seventh part but greater than ten seventy-first parts.[a]

Prop. 2

The circle bears to the square on the diameter the ratio 11 : 14.

Let there be a circle with diameter AB, and let the square ΓH be circumscribed, and let $\Delta E = 2\Gamma\Delta$, $EZ = \frac{1}{7}\Gamma\Delta$. Then, since $A\Gamma E : A\Gamma\Delta = 21 : 7$, while $A\Gamma\Delta : AEZ = 7 : 1$ [Euclid vi. 1], it follows that $A\Gamma Z : A\Gamma\Delta = 22 : 7$.[b] But the square $\Gamma H = 4\, A\Gamma\Delta$, while the triangle AΓΔZ is equal to the circle AB; therefore the circle bears to the square ΓH the ratio 11 : 14.

[a] We know from Heron, *Metrica* i. 26 (ed. Schöne 66. 13-17), that Archimedes made a still closer approximation to π. The figures in the Greek text are unfortunately corrupt, but a plausible correction by Heiben (*Nordisk Tidsskrift for Filologi*, 3e Sér. xx. Fasc. 1-2) would give the approximation

$$3{\cdot}141697\ldots > \pi > 3{\cdot}141495\ldots$$

Ptolemy, *Syntaxis* vi. 7 (ed. Heiberg 513. 1-5), gives the value of π in sexagesimal fractions as $3 + \frac{8}{60} + \frac{30}{60^2}$ or $3{\cdot}141\dot{6}$.

[b] For ἀνάπαλιν $AEZ : A\Gamma\Delta = 1 : 7$, and $A\Gamma E : A\Gamma\Delta = 21 : 7$, and therefore συνθέντι $A\Gamma Z : A\Gamma\Delta = (AEZ + A\Gamma E) : A\Gamma\Delta = 22 : 7$. But the same result could be obtained immediately from Eucl. vi. 1.

---

[1] "Hic locus ἐπεὶ . . . δειχθήσεται mire confusus transcriptori tribuendus, qui eum addidit, postquam prop. 2 et 3 permutavit; neque enim Archimedes hanc propositionem ante prop. 3, qua nititur, posuit" (Heiberg).

(c) Solutions by Higher Curves

(i.) *General*

Simpl. *in Cat.* 7, ed. Kalbfleisch 192. 15-25

Ἔστιν δὲ τετραγωνισμὸς κύκλου, ὅταν τῷ δοθέντι κύκλῳ ἴσον τετράγωνον συστησώμεθα. τοῦτο δὲ Ἀριστοτέλης μέν, ὡς ἔοικεν, οὔπω ἐγνώκει, παρὰ δὲ τοῖς Πυθαγορείοις ηὑρῆσθαί φησιν Ἰάμβλιχος, "ὡς δῆλόν ἐστιν ἀπὸ τῶν Σέξτου τοῦ Πυθαγορείου ἀποδείξεων, ὃς ἄνωθεν κατὰ διαδοχὴν παρέλαβεν τὴν μέθοδον τῆς ἀποδείξεως. καὶ ὕστερον δέ, φησίν, Ἀρχιμήδης διὰ τῆς Λυκομήδους[1] γραμμῆς καὶ Νικομήδης διὰ τῆς ἰδίως τετραγωνιζούσης καλουμένης καὶ Ἀπολλώνιος διά τινος γραμμῆς, ἣν αὐτὸς μὲν κοχλιοειδοῦς ἀδελφὴν προσαγορεύει, ἡ αὐτὴ δέ ἐστιν τῇ Νικομήδους, καὶ Κάρπος δὲ διά τινος γραμμῆς, ἣ ἁπλῶς ἐκ διπλῆς κινήσεως καλεῖ, ἄλλοι τε πολλοὶ ποικίλως τὸ πρόβλημα κατεσκεύασαν," ὡς Ἰάμβλιχος ἱστορεῖ.

[1] No meaning can be extracted from Λυκομήδους, which is an otherwise unknown word. The correct reading is probably ἑλικοειδοῦς, "spiral-shaped."

(c) Solutions by Higher Curves

(i.) *General*

Simplicius, *Commentary on Aristotle's Categories* 7, ed. Kalbfleisch 192. 15-25

The circle is squared when we construct a square equal to the given circle. Aristotle, it would appear, did not know how to do this, but Iamblichus says it was discovered by the Pythagoreans, "as is plain from the proofs of Sextus the Pythagorean,[a] who received the method of the proof from early tradition. And later (he says), Archimedes effected it by means of the spiral-shaped curve,[b] Nicomedes by means of the curve known by the special name *quadratrix*, Apollonius by means of a certain curve which he himself calls *sister of the cochloid*, but which is the same as Nicomedes' curve,[c] Carpus by means of a certain curve which he simply calls that *arising from a double motion*,[d] and many others constructed a solution of this problem in divers ways," as Iamblichus relates.

[a] Sextus (more properly Sextius) lived in the reign of Augustus (or Tiberius) and there is no valid reason for believing the early Pythagoreans solved the problem.

[b] Archimedes himself in his book *On Spirals*, which will be noticed when we come to him, merely uses the spiral to rectify the circle (Prop. 19). But the quadrature follows from *Measurement of a Circle*, Prop. 1.

[c] Nothing further is known of Apollonius's "sister of the *cochloid*," but Heath (*H.G.M.* i. 232) points out that Apollonius wrote a treatise on the *cochlias*, or cylindrical helix, that the subtangent to this curve can be used to square the circular section of the cylinder, and that the name is sufficiently akin to justify Apollonius in speaking of it as the "sister of the *cochloid*."

[d] Tannery thought this was the cycloid, but there is no evidence.

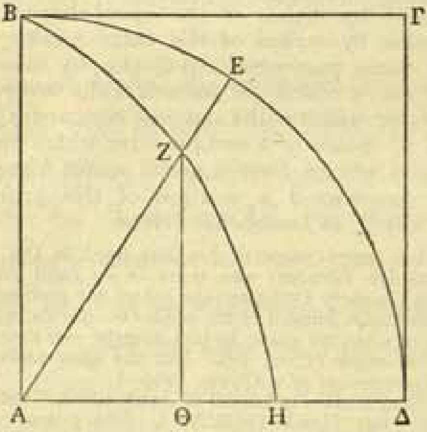
(ii.) *The Quadratrix*

Papp. *Coll.* iv. 30. 45–32. 50, ed. Hultsch 250. 33–258. 19

Construction of the Curve

λʹ. Εἰς τὸν τετραγωνισμὸν τοῦ κύκλου παρελήφθη τις ὑπὸ Δεινοστράτου καὶ Νικομήδους γραμμὴ καί τινων ἄλλων νεωτέρων ἀπὸ τοῦ περὶ αὐτὴν συμπτώματος λαβοῦσα τοὔνομα· καλεῖται γὰρ ὑπ' αὐτῶν τετραγωνίζουσα καὶ γένεσιν ἔχει τοιαύτην.

Ἐκκείσθω τετράγωνον τὸ ΑΒΓΔ καὶ περὶ κέντρον τὸ Α περιφέρεια γεγράφθω ἡ ΒΕΔ, καὶ



κινείσθω ἡ μὲν ΑΒ οὕτως ὥστε τὸ μὲν Α σημεῖον μένειν τὸ δὲ Β φέρεσθαι κατὰ τὴν ΒΕΔ περιφέρειαν, ἡ δὲ ΒΓ παράλληλος ἀεὶ διαμένουσα τῇ ΑΔ τῷ Β σημείῳ φερομένῳ[1] κατὰ τῆς ΒΑ συνακολουθείτω, καὶ ἐν ἴσῳ χρόνῳ ἥ τε ΑΒ κινουμένη

(ii.) *The Quadratrix*

Pappus, *Collection* iv. 30. 45–32. 50, ed. Hultsch 250. 33–258. 19

Construction of the Curve

30. For the squaring of the circle a certain line was used by Dinostratus and Nicomedes and certain other more recent geometers, and it takes its name from its special property; for it is called by them the quadratrix,[a] and it is generated in this way.

Let ΑΒΓΔ be a square, and with centre Α let the arc ΒΕΔ be described, and let ΑΒ be so moved that the point Α remains fixed while Β is carried along the arc ΒΕΔ; furthermore let ΒΓ, while always remaining parallel to ΑΔ, follow the point Β in its motion along ΒΑ, and in equal times let ΑΒ, moving uni-

[a] Heath (*H.G.M.* i. 225-226) shows that the quadratrix was discovered by Hippias and that he may himself have used it (though this is not absolutely certain) to rectify, and so to square, the circle.

---

[1] *σημεῖον φέρον ἐν ᾧ* in the MSS. was corrected by Torelli.

ὁμαλῶς τὴν ὑπὸ ΒΑΔ γωνίαν, τουτέστιν τὸ Β σημεῖον τὴν ΒΕΔ περιφέρειαν, διανυέτω, καὶ ἡ ΒΓ τὴν ΒΑ εὐθεῖαν παροδευέτω, τουτέστιν τὸ Β σημεῖον κατὰ τῆς ΒΑ φερέσθω. συμβήσεται δῆλον τῇ ΑΔ εὐθείᾳ ἅμα ἐφαρμόζειν ἑκατέραν τήν τε ΑΒ καὶ τὴν ΒΓ. τοιαύτης δὴ γινομένης κινήσεως τεμοῦσιν ἀλλήλας ἐν τῇ φορᾷ αἱ ΒΓ, ΒΑ εὐθεῖαι κατά τι σημεῖον αἰεὶ συμμεθιστάμενον αὐταῖς, ὑφ' οὗ σημείου γράφεταί τις ἐν τῷ μεταξὺ τόπῳ τῶν τε ΒΑΔ εὐθειῶν καὶ τῆς ΒΕΔ περιφερείας γραμμὴ ἐπὶ τὰ αὐτὰ κοίλη, οἵα ἐστὶν ἡ ΒΖΗ, ἣ καὶ χρειώδης εἶναι δοκεῖ πρὸς τὸ τῷ δοθέντι κύκλῳ τετράγωνον ἴσον εὑρεῖν. τὸ δὲ ἀρχικὸν αὐτῆς σύμπτωμα τοιοῦτόν ἐστιν. ἥτις γὰρ ἂν διαχθῇ τυχοῦσα ⟨πρὸς τὴν περιφέρειαν, ὡς ἡ ΑΖΕ, ἔσται ὡς ὅλη ἡ⟩[1] περιφέρεια πρὸς τὴν ΕΔ, ἡ ΒΑ εὐθεῖα πρὸς τὴν ΖΘ· τοῦτο γὰρ ἐκ τῆς γενέσεως τῆς γραμμῆς φανερόν ἐστιν.

Sporus's Criticisms

λα'. Δυσαρεστεῖται δὲ αὐτῇ ὁ Σπόρος εὐλόγως διὰ ταῦτα. πρῶτον μὲν γὰρ πρὸς ὃ δοκεῖ χρειώδης εἶναι πρᾶγμα, τοῦτ' ἐν ὑποθέσει λαμβάνει. πῶς γὰρ δυνατόν, δύο σημείων ἀρξαμένων ἀπὸ τοῦ Β

[1] πρὸς τὴν . . . ὅλη ἡ add. Hultsch.

formly, pass through the angle BAΔ (that is, the point B pass along the arc BΔ), and BΓ pass by the straight line BA (that is, let the point B traverse the length of BA). Plainly then both AB and BΓ will coincide simultaneously with the straight line AΔ. While the motion is in progress the straight lines BΓ, BA will cut one another in their movement at a certain point which continually changes place with them, and by this point there is described in the space between the straight lines BA, AΔ and the arc BEΔ a concave curve, such as BZH, which appears to be serviceable for the discovery of a square equal to the given circle. Its principal property is this. If any straight line, such as AZE, be drawn to the circumference, the ratio of the whole arc to EΔ will be the same as the ratio of the straight line BA to ZΘ; for this is clear from the manner in which the line was generated.[a]

Sporus's Criticisms [b]

31. With this Sporus is rightly displeased for these reasons. In the first place, the end for which the construction seems to be useful is assumed in the hypothesis. For how is it possible, with two points

[a] If $AZ=\rho$, $\angle Z A \Delta=\phi$, $AB=a$, then the equation of the curve is

$$\frac{\frac{1}{2}\pi}{\phi}=\frac{a}{\rho\sin\phi}$$

or

$$\pi\rho\sin\phi=2a\phi.$$

[b] These acute criticisms of the quadratrix as a practical method of squaring the circle appear to be well founded. Sporus, who was not much older than Pappus himself, lived towards the end of the third century A.D. He compiled a work called Κηρία giving extracts on the quadrature of the circle and duplication of the cube.

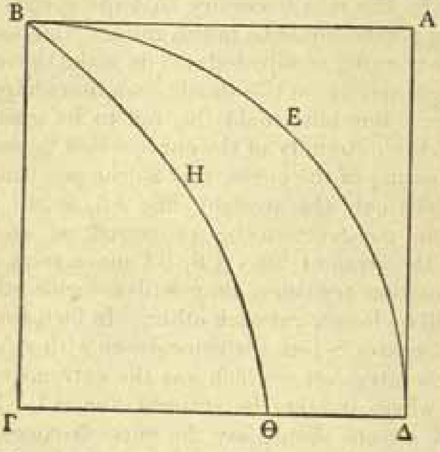
κινεῖσθαι, τὸ μὲν κατ' εὐθείας ἐπὶ τὸ Α, τὸ δὲ κατὰ περιφερείας ἐπὶ τὸ Δ ἐν ἴσῳ χρόνῳ συναποκαταστῆσαι[1] μὴ πρότερον τὸν λόγον τῆς ΑΒ εὐθείας πρὸς τὴν ΒΕΔ περιφέρειαν ἐπιστάμενον; ἐν γὰρ τούτῳ τῷ λόγῳ καὶ τὰ τάχη τῶν κινήσεων ἀνάγκη εἶναι. ἐπεὶ πῶς οἷόν τε συναποκαταστῆναι τάχεσιν ἀκρίτοις χρώμενα, πλὴν εἰ μὴ ἂν κατὰ τύχην ποτὲ συμβῇ; τοῦτο δὲ πῶς οὐκ ἄλογον; ἔπειτα δὲ τὸ πέρας αὐτῆς ᾧ χρῶνται πρὸς τὸν τετραγωνισμὸν τοῦ κύκλου, τουτέστιν καθ' ὃ τέμνει σημεῖον τὴν ΑΔ εὐθεῖαν, οὐχ εὑρίσκεται. νοείσθω δὲ ἐπὶ τῆς προκειμένης τὰ λεγόμενα καταγραφῆς· ὁπόταν γὰρ αἱ ΓΒ, ΒΑ φερόμεναι συναποκατασταθῶσιν, ἐφαρμόσουσιν τῇ ΑΔ καὶ τομὴν οὐκέτι ποιήσουσιν ἐν ἀλλήλαις· παύεται γὰρ ἡ τομὴ πρὸ τῆς ἐπὶ τὴν ΑΔ ἐφαρμογῆς ἥπερ τομὴ πέρας αὖ ἐγένετο τῆς γραμμῆς, καθ' ὃ τῇ ΑΔ εὐθείᾳ συνέπιπτεν. πλὴν εἰ μὴ λέγοι τις ἐπινοεῖσθαι προσεκβαλλομένην τὴν γραμμήν, ὡς ὑποτιθέμεθα τὰς εὐθείας, ἕως τῆς ΑΔ. τοῦτο δ' οὐχ ἕπεται ταῖς ὑποκειμέναις ἀρχαῖς, ἀλλ' ὡς ἂν ληφθείη τὸ Η σημεῖον προειλημμένου τοῦ τῆς περιφερείας πρὸς τὴν εὐθεῖαν λόγου. χωρὶς δὲ τοῦ δοθῆναι τὸν λόγον τοῦτον οὐ χρὴ τῇ τῶν εὑρόντων ἀνδρῶν δόξῃ πιστεύοντας παραδέχεσθαι τὴν γραμμὴν μηχανικωτέραν πως οὖσαν [καὶ εἰς πολλὰ προβλήματα χρησιμεύουσαν τοῖς μηχανικοῖς].[2] ἀλλὰ πρότερον παραδεκτέον ἐστὶ τὸ δι' αὐτῆς δεικνύμενον πρόβλημα.

[1] συναποκαταστῆσαι coniecit Hultsch.
[2] καὶ . . . μηχανικοῖς interpolatori tribuit Hultsch.

beginning to move from B, to make one of them move along a straight line to A and the other along a circumference to Δ in equal time unless first the ratio of the straight line AB to the circumference BEΔ is known? For it is necessary that the speeds of the moving points should be in this ratio. And how then could one, using unadjusted speeds, make the motions end together, unless this should sometimes happen by chance? But how could this fail to be irrational? Again, the extremity of the curve which they use for the squaring of the circle, that is, the point in which the curve cuts the straight line AΔ, is not found. Let the construction be conceived as aforesaid. When the straight lines ΓB, BA move so as to end their motion together, they will coincide with AΔ and will no longer cut each other. In fact, the intersection ceases before the coincidence with AΔ, yet it was this intersection which was the extremity of the curve where it met the straight line AΔ. Unless, indeed, anyone should say the curve is conceived as produced, in the same way that we produce straight lines, as far as AΔ. But this does not follow from the assumptions made; the point H can be found only by assuming the ratio of the circumference to the straight line. So unless this ratio is given, we must beware lest, in following the authority of those men who discovered the line, we admit its construction, which is more a matter of mechanics. But first let us deal with that problem which we have said can be proved by means of it.

Application of Quadratrix to Squaring of Circle

Τετραγώνου γὰρ ὄντος τοῦ ΑΒΓΔ καὶ τῆς μὲν περὶ τὸ κέντρον τὸ Γ περιφερείας τῆς ΒΕΔ, τῆς



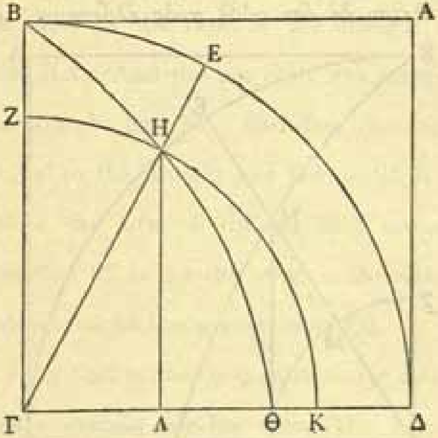
δὲ ΒΗΘ τετραγωνιζούσης γινομένης, ὡς προείρηται, δείκνυται, ὡς ἡ ΔΕΒ περιφέρεια πρὸς τὴν ΒΓ εὐθεῖαν, οὕτως ἡ ΒΓ πρὸς τὴν ΓΘ εὐθεῖαν. εἰ γὰρ μή ἐστιν, ἤτοι πρὸς μείζονα ἔσται τῆς ΓΘ ἢ πρὸς ἐλάσσονα.

Ἔστω πρότερον, εἰ δυνατόν, πρὸς μείζονα τὴν ΓΚ, καὶ περὶ κέντρον τὸ Γ περιφέρεια ἡ ΖΗΚ γεγράφθω τέμνουσα τὴν γραμμὴν κατὰ τὸ Η, καὶ κάθετος ἡ ΗΛ, καὶ ἐπιζευχθεῖσα ἡ ΓΗ ἐκβεβλήσθω ἐπὶ τὸ Ε. ἐπεὶ οὖν ἐστιν ὡς ἡ ΔΕΒ περιφέρεια πρὸς τὴν ΒΓ εὐθεῖαν, οὕτως ἡ ΒΓ,

Application of Quadratrix to Squaring of Circle

If ΑΒΓΔ is a square and ΒΕΔ the arc of a circle with centre Γ, while ΒΗΘ is a quadratrix generated in the aforesaid manner, it is proved that the ratio of the arc ΔΕΒ towards the straight line ΒΓ is the same as that of ΒΓ towards the straight line ΓΘ. For if it is not, the ratio of the arc ΔΕΒ towards the straight line ΒΓ will be the same as that of ΒΓ towards either a straight line greater than ΓΘ or a straight line less than ΓΘ.

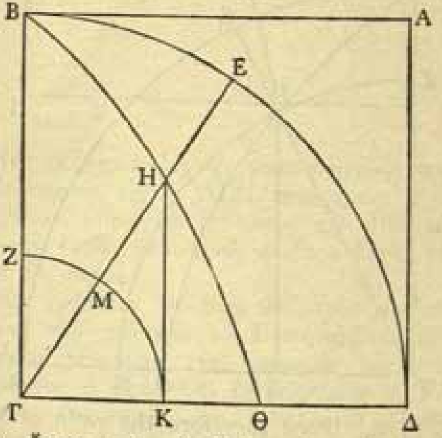
Let it be the former, if possible, towards a greater straight line ΓΚ, and with centre Γ let the arc ΖΗΚ be drawn cutting the curve at Η, and let the perpendicular ΗΛ be drawn, and let ΓΗ be joined and pro-



duced to Ε. Since therefore the ratio of the arc ΔΕΒ towards the straight line ΒΓ is the same as the

τουτέστιν ἡ ΓΔ, πρὸς τὴν ΓΚ, ὡς δὲ ἡ ΓΔ πρὸς τὴν ΓΚ, ἡ ΒΕΔ περιφέρεια πρὸς τὴν ΖΗΚ περιφέρειαν (ὡς γὰρ ἡ διάμετρος τοῦ κύκλου πρὸς τὴν διάμετρον, ἡ περιφέρεια τοῦ κύκλου πρὸς τὴν περιφέρειαν), φανερὸν ὅτι ἴση ἐστὶν ἡ ΖΗΚ περιφέρεια τῇ ΒΓ εὐθείᾳ. καὶ ἐπειδὴ διὰ τὸ σύμπτωμα τῆς γραμμῆς ἐστιν ὡς ἡ ΒΕΔ περιφέρεια πρὸς τὴν ΕΔ, οὕτως ἡ ΒΓ πρὸς τὴν ΗΛ, καὶ ὡς ἄρα ἡ ΖΗΚ πρὸς τὴν ΗΚ περιφέρειαν, οὕτως ἡ ΒΓ εὐθεῖα πρὸς τὴν ΗΛ. καὶ ἐδείχθη ἴση ἡ ΖΗΚ περιφέρεια τῇ ΒΓ εὐθείᾳ· ἴση ἄρα καὶ ἡ ΗΚ περιφέρεια τῇ ΗΛ εὐθείᾳ, ὅπερ ἄτοπον. οὐκ ἄρα ἐστὶν ὡς ἡ ΒΕΔ περιφέρεια πρὸς τὴν ΒΓ εὐθεῖαν, οὕτως ἡ ΒΓ πρὸς μείζονα τῆς ΓΘ.

λβʹ. Λέγω δὲ ὅτι οὐδὲ πρὸς ἐλάσσονα. εἰ γὰρ



δυνατόν, ἔστω πρὸς τὴν ΚΓ, καὶ περὶ κέντρον τὸ

ratio of ΒΓ, that is ΓΔ, towards ΓΚ, and the ratio of ΓΔ towards ΓΚ is the same as that of the arc ΒΕΔ towards the arc ΖΗΚ (for the arcs of circles are in the same ratio as their diameters), it is clear that the arc ΖΗΚ is equal to the straight line ΒΓ. And since by the property of the curve the ratio of the arc ΒΕΔ towards ΕΔ is the same as the ratio of ΒΓ towards ΗΛ, therefore the ratio of ΖΗΚ towards the arc ΗΚ is the same as the ratio of the straight line ΒΓ towards ΗΛ. And the arc ΖΗΚ was proved equal to the straight line ΒΓ; therefore the arc ΗΚ is also equal to the straight line ΗΛ, which is absurd. Therefore the ratio of the arc ΒΕΔ towards the straight line ΒΓ is not the same as the ratio of ΒΓ towards a straight line greater than ΓΘ.

32. I say that neither is it equal to the ratio of ΒΓ towards a straight line less than ΓΘ. For, if it is possible, let the ratio be towards ΚΓ, and with centre

Γ περιφέρεια γεγράφθω ἡ ΖΜΚ, καὶ πρὸς ὀρθὰς τῇ ΓΔ ἡ ΚΗ τέμνουσα τὴν τετραγωνίζουσαν κατὰ τὸ Η, καὶ ἐπιζευχθεῖσα ἡ ΓΗ ἐκβεβλήσθω ἐπὶ τὸ Ε. ὁμοίως δὴ τοῖς προγεγραμμένοις δείξομεν καὶ τὴν ΖΜΚ περιφέρειαν τῇ ΒΓ εὐθείᾳ ἴσην, καὶ ὡς τὴν ΒΕΔ περιφέρειαν πρὸς τὴν ΕΔ, τουτέστιν ὡς τὴν ΖΜΚ πρὸς τὴν ΜΚ, οὕτως τὴν ΒΓ εὐθεῖαν πρὸς τὴν ΗΚ. ἐξ ὧν φανερὸν ὅτι ἴση ἔσται ἡ ΜΚ περιφέρεια τῇ ΚΗ εὐθείᾳ, ὅπερ ἄτοπον. οὐκ ἄρα ἔσται ὡς ἡ ΒΕΔ περιφέρεια πρὸς τὴν ΒΓ εὐθεῖαν, οὕτως ἡ ΒΓ πρὸς ἐλάσσονα τῆς ΓΘ. ἐδείχθη δὲ ὅτι οὐδὲ πρὸς μείζονα· πρὸς αὐτὴν ἄρα τὴν ΓΘ.

Ἔστι δὲ καὶ τοῦτο φανερὸν ὅτι ἡ τῶν ΘΓ, ΓΒ εὐθειῶν τρίτη ἀνάλογον λαμβανομένη εὐθεῖα ἴση ἔσται τῇ ΒΕΔ περιφερείᾳ, καὶ ἡ τετραπλασίων αὐτῆς τῇ τοῦ ὅλου κύκλου περιφερείᾳ. εὑρημένης δὲ τῇ τοῦ κύκλου περιφερείᾳ ἴσης εὐθείας πρόδηλον ὡς δὴ καὶ αὐτῷ τῷ κύκλῳ ῥᾴδιον ἴσον τετράγωνον συστήσασθαι· τὸ γὰρ ὑπὸ τῆς περιμέτρου τοῦ κύκλου καὶ τῆς ἐκ τοῦ κέντρου διπλάσιόν ἐστι τοῦ κύκλου, ὡς Ἀρχιμήδης ἀπέδειξεν.

3. TRISECTION OF AN ANGLE

(a) Types of Geometrical Problems

Papp. *Coll.* iv. 36. 57-59, ed. Hultsch 270. 1–272. 14

λϛʹ. Τὴν δοθεῖσαν γωνίαν εὐθύγραμμον εἰς τρία ἴσα τεμεῖν οἱ παλαιοὶ γεωμέτραι θελήσαντες ἠπόρησαν δι' αἰτίαν τοιαύτην. τρία γένη φαμὲν εἶναι

Γ let the arc ZMK be described, and let KH at right angles to ΓΔ cut the quadratrix at H, and let ΓH be joined and produced to E. In similar manner to what has been written above, we shall prove also that the arc ZMK is equal to the straight line BΓ, and that the ratio of the arc BEΔ towards EΔ, that is, the ratio of ZMK towards MK, is the same as that of the straight line BΓ towards HK. From this it is clear that the arc MK is equal to the straight line KH, which is absurd. The ratio of the arc BEΔ towards the straight line BΓ is therefore not the same as the ratio of BΓ towards a straight line less than ΓΘ. Moreover it was proved not the same as the ratio of BΓ towards a straight line greater than ΓΘ; therefore it is the same as the ratio of BΓ towards ΓΘ itself.

This also is clear, that if a straight line is taken as a third proportional to the straight lines ΘΓ, ΓB it will be equal to the arc BEΔ, and four times this straight line will be equal to the circumference of the whole circle. A straight line equal to the circumference of the circle having been found, a square can easily be constructed equal to the circle itself. For the rectangle contained by the perimeter of the circle and the radius is double of the circle, as Archimedes demonstrated.[a]

3. TRISECTION OF AN ANGLE

(a) Types of Geometrical Problems

Pappus, *Collection* iv. 36. 57-59, ed. Hultsch 270. 1–272. 14

36. When the ancient geometers sought to divide a given rectilineal angle into three equal parts they were at a loss for this reason. We say that there

[a] See *supra*, pp. 316-321.

τῶν ἐν γεωμετρίᾳ προβλημάτων, καὶ τὰ μὲν αὐτῶν ἐπίπεδα καλεῖσθαι, τὰ δὲ στερεά, τὰ δὲ γραμμικά. τὰ μὲν οὖν δι' εὐθείας καὶ κύκλου περιφερείας δυνάμενα λύεσθαι λέγοιτ' ἂν εἰκότως ἐπίπεδα· καὶ γὰρ αἱ γραμμαὶ δι' ὧν εὑρίσκεται τὰ τοιαῦτα προβλήματα τὴν γένεσιν ἔχουσιν ἐν ἐπιπέδῳ. ὅσα δὲ λύεται προβλήματα παραλαμβανομένης εἰς τὴν εὕρεσιν μιᾶς τῶν τοῦ κώνου τομῶν ἢ καὶ πλειόνων, στερεὰ ταῦτα κέκληται· πρὸς γὰρ τὴν κατασκευὴν χρήσασθαι στερεῶν σχημάτων ἐπιφανείαις, λέγω δὲ ταῖς κωνικαῖς, ἀναγκαῖον. τρίτον δέ τι προβλημάτων ὑπολείπεται γένος τὸ καλούμενον γραμμικόν· γραμμαὶ γὰρ ἕτεραι παρὰ τὰς εἰρημένας εἰς τὴν κατασκευὴν λαμβάνονται ποικιλωτέραν ἔχουσαι τὴν γένεσιν καὶ βεβιασμένην μᾶλλον, ἐξ ἀτακτοτέρων ἐπιφανειῶν καὶ κινήσεων ἐπιπεπλεγμένων γεννώμεναι. τοιαῦται δέ εἰσιν αἵ τε ἐν τοῖς πρὸς ἐπιφανείαις καλουμένοις τόποις εὑρισκόμεναι γραμμαὶ ἕτεραί τε τούτων ποικιλώτεραι καὶ πολλαὶ τὸ πλῆθος ὑπὸ Δημητρίου τοῦ Ἀλεξανδρέως ἐν ταῖς Γραμμικαῖς ἐπιστάσεσι καὶ Φίλωνος τοῦ Τυανέως ἐξ ἐπιπλοκῆς πλεκτοειδῶν τε καὶ ἑτέρων παντοίων ἐπιφανειῶν εὑρισκόμεναι πολλὰ καὶ θαυμαστὰ συμπτώματα περὶ αὐτὰς ἔχουσαι. καί τινες αὐτῶν ὑπὸ τῶν νεωτέρων ἠξιώθησαν λόγου πλείονος, μία δέ τις ἐξ αὐτῶν ἐστιν ἡ καὶ παράδοξος ὑπὸ τοῦ Μενελάου κληθεῖσα γραμμή. τοῦ δὲ αὐτοῦ γένους ἕτεραι ἑλικές εἰσιν

[a] Whether τόποι πρὸς ἐπιφανείαις are "loci which are surfaces" or "loci which lie on surfaces" (*e.g.*, the cylindrical helix) is a moot point. Euclid wrote two books under the title.

are three kinds of problems in geometry, some being called *plane*, some *solid*, some *linear*. Those which can be solved by means of a straight line and a circumference of a circle are properly called plane; for the lines by which such problems are solved have their origin in a plane. Such problems, however, as are solved by using for their discovery one or more of the sections of the cone are called solid; for in the construction it is necessary to use surfaces of solid figures, I mean the conic surfaces. There remains a third kind of problem called linear; for other lines besides those mentioned are used for their construction, having a more complicated and less natural origin as they are generated from more irregular surfaces and intricate movements. Among such lines are those found in the so-called *surface-loci*,[a] and many others more complicated than these were discovered by Demetrius of Alexandria in his *Linear Considerations* and Philon of Tyana[b] as a result of interweaving plektoids and other surfaces of all kinds, and they exhibit many wonderful properties. Some of these curves were investigated more fully by more recent geometers, and among them in the line called *paradoxical* by Menelaus.[c] Other lines of

[b] Nothing further is known of these writers, unless Demetrius be the Cynic, mentioned by Diogenes Laertius, who lived about 300 B.C., or the philosopher who flourished in the time of Seneca.

[c] Menelaus flourished *c.* A.D. 100 and his name is preserved in a famous theorem in spherical trigonometry. Tannery (*Mémoires scientifiques* ii. p. 17) has suggested that the curve called *paradoxical* was Viviani's curve of double curvature, defined as the intersection of a sphere with a cylinder touching it internally and having for its diameter the radius of the sphere. It is a particular case of Eudoxus's *hippopede* (see *infra*, p. 414), and the portion lying outside

τετραγωνίζουσαί τε καὶ κοχλοειδεῖς καὶ κισσοειδεῖς. δοκεῖ δέ πως ἁμάρτημα τὸ τοιοῦτον οὐ μικρὸν εἶναι τοῖς γεωμέτραις, ὅταν ἐπίπεδον πρόβλημα διὰ τῶν κωνικῶν ἢ τῶν γραμμικῶν ὑπό τινος εὑρίσκηται, καὶ τὸ σύνολον ὅταν ἐξ ἀνοικείου λύηται γένους, οἷόν ἐστιν τὸ ἐν τῷ πέμπτῳ τῶν Ἀπολλωνίου Κωνικῶν ἐπὶ τῆς παραβολῆς πρόβλημα καὶ ἡ ἐν τῷ περὶ τῆς ἕλικος ὑπὸ Ἀρχιμήδους λαμβανομένη στερεοῦ νεῦσις ἐπὶ κύκλον· μηδενὶ

---

the curve of the surface of the hemisphere on which it lies is equal to the square on the diameter of the sphere; the fact that this area can be squared is thought to justify the name *paradoxical*. An Arabian tradition that Menelaus reproduced in his *Elements of Geometry* Archytas's solution of the problem of duplicating the cube (involving the intersection of a tore, cylinder and cone) lends a certain plausibility to the suggestion (*v.* Heath, *H.G.M.* ii. 261, Loria, *Le scienze esatte*, pp. 518-520).

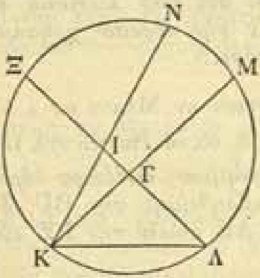
[a] Heath identifies this (*Apollonius of Perga* cxxvii-cxxix) as *Conics* v. 58, where Apollonius finds the feet of the normals to a parabola passing through a given point by constructing a rectangular hyperbola whose intersections with the parabola give the required points. The feet of the normals could be found in the case of the parabola (though not of the ellipse or hyperbola) by the intersection of the parabola with a certain *circle*.

[b] The assumption made by Archimedes (Περὶ ἑλίκων 8, 9) is to the following effect, the relevant portion of his figure being detached:

If ΞΛ, KM are two chords of a circle, meeting at right angles at Γ, so that ΞΓ > ΓΛ, then it is possible to draw another chord KN meeting ΞΛ in I such that IN = MΓ (or, as Archimedes expresses the matter, *it is possible to place the straight line* IN *equal to* MΓ *and verging towards* K).

this kind are spirals and quadratices and cochloids and cissoids. It appears to be no small error for geometers when a plane problem is solved by conics or other curved lines, and in general when any problem is solved by an inappropriate kind, as in the problem concerning the parabola in the fifth book of the *Conics* of Apollonius [a] and the verging of a solid character with respect to a circle assumed by Archimedes in his book on the spiral [b]; for it is possible

In general, the line KN is determined by the intersection of a hyperbola and a parabola, as Pappus himself shows in



another place (iv. 52-53, ed. Hultsch 298-302). The particular case where ΞΛ is a diameter bisecting the chord KM in Γ can be solved by plane methods, namely, by the "application of areas"; the solution for the case where IN is to be made equal to $\sqrt{\frac{5}{4}}$ (radius of the circle) is assumed by Hippocrates in the fragment from Eudemus preserved by Simplicius (see *supra*, p. 244 n. *a*).

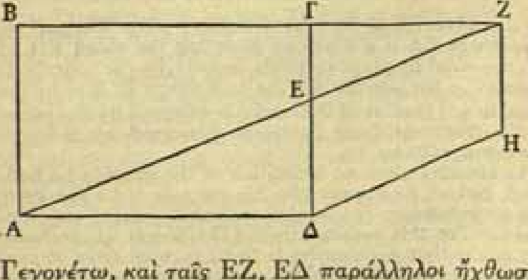
Archimedes gives no indication of the solution he had in mind, but all he requires for his purpose is its *possibility*; and its possibility can be demonstrated without any use of conics. For this reason Heath (*The Works of Archimedes* civ) thinks that Archimedes is to be excused from Pappus's censure that he had solved a plane problem by solid methods.

γὰρ προσχρώμενον στερεῷ δυνατὸν εὑρεῖν τὸ ὑπ' αὐτοῦ γραφόμενον θεώρημα, λέγω δὴ τὸ τὴν περιφέρειαν τοῦ ἐν τῇ πρώτῃ περιφορᾷ κύκλου ἴσην ἀποδεῖξαι τῇ πρὸς ὀρθὰς ἀγομένῃ εὐθείᾳ τῇ ἐκ τῆς γενέσεως ἕως τῆς ἐφαπτομένης τῆς ἕλικος. τοιαύτης δὴ τῆς διαφορᾶς τῶν προβλημάτων ὑπαρχούσης οἱ πρότεροι γεωμέτραι τὸ προειρημένον ἐπὶ τῆς γωνίας πρόβλημα τῇ φύσει στερεὸν ὑπάρχον διὰ τῶν ἐπιπέδων ζητοῦντες οὐχ οἷοί τ' ἦσαν εὑρίσκειν· οὐδέπω γὰρ αἱ τοῦ κώνου τομαὶ συνήθεις ἦσαν αὐτοῖς, καὶ διὰ τοῦτο ἠπόρησαν· ὕστερον μέντοι διὰ τῶν κωνικῶν ἐτριχοτόμησαν τὴν γωνίαν εἰς τὴν εὕρεσιν χρησάμενοι τῇ ὑπογεγραμμένῃ νεύσει.

(b) Solution by Means of a Verging

Ibid. iv. 36. 60, ed. Hultsch 272. 15–274. 2

Παραλληλογράμμου δοθέντος ὀρθογωνίου τοῦ ΑΒΓΔ καὶ ἐκβληθείσης τῆς ΒΓ, δέον ἔστω διαγαγόντα τὴν ΑΕ ποιεῖν τὴν ΕΖ εὐθεῖαν ἴσην τῇ δοθείσῃ.



Γεγονέτω, καὶ ταῖς ΕΖ, ΕΔ παράλληλοι ἤχθωσαν

without using anything solid to find the theorem stated by him, I mean the theorem proving that the circumference of the circle in the first turn is equal to the straight line drawn at right angles to the initial line to meet the tangent to the spiral.[a] Since problems differ in this way, the earlier geometers were not able to solve the aforementioned problem about the angle, when they sought to do so by means of planes, because it is by nature solid; for they were not yet familiar with the sections of the cone, and for this reason were at a loss. Later, however, they trisected the angle by means of the conics, using in the solution the verging described below.

(b) Solution by Means of a Verging

Ibid. iv. 36. 60, ed. Hultsch 272. 15–274. 2

Given a right-angled[b] parallelogram ΑΒΓΔ, with ΒΓ produced, let it be required to draw ΑΕ so as to make the straight line ΕΖ equal to the given straight line.

Suppose it done, and let ΔΗ, ΗΖ be drawn parallel

[a] Archimedes' enunciation (Περὶ ἑλίκων 18) is: Εἴ κα τᾶς ἕλικος τᾶς ἐν τᾷ πρώτᾳ περιφορᾷ γεγραμμένας εὐθεῖα γραμμὰ ἐπιψαύῃ κατὰ τὸ πέρας τᾶς ἕλικος, ἀπὸ δὲ τοῦ σαμείου, ὅ ἐστιν ἀρχὰ τᾶς ἕλικος, ποτ' ὀρθὰς ἀχθῇ τις τᾷ ἀρχᾷ τᾶς περιφορᾶς, ἁ ἀχθεῖσα συμπεσεῖται τᾷ ἐπιψαυούσᾳ, καὶ ἁ μεταξὺ εὐθεῖα τᾶς ἐπιψαυούσας καὶ τᾶς ἀρχᾶς τᾶς ἕλικος ἴσα ἐσσεῖται τᾷ τοῦ πρώτου κύκλου περιφερείᾳ.

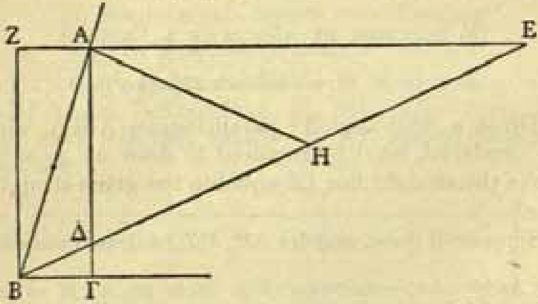
[b] It is not, in fact, necessary that the parallelogram should be right-angled.

αἱ ΔΗ, ΗΖ. ἐπεὶ οὖν δοθεῖσά ἐστιν ἡ ΖΕ καὶ ἔστιν ἴση τῇ ΔΗ, δοθεῖσα ἄρα καὶ ἡ ΔΗ. καὶ δοθὲν τὸ Δ· τὸ Η ἄρα πρὸς θέσει κύκλου περιφερείᾳ. καὶ ἐπεὶ τὸ ὑπὸ ΒΓΔ δοθὲν καὶ ἔστιν ἴσον τῷ ὑπὸ ΒΖ, ΕΔ, δοθὲν ἄρα καὶ τὸ ὑπὸ ΒΖ ΕΔ, τουτέστιν τὸ ὑπὸ ΒΖΗ· τὸ Η ἄρα πρὸς ὑπερβολῇ. ἀλλὰ καὶ πρὸς θέσει κύκλου περιφερείᾳ· δοθὲν ἄρα τὸ Η.

Ibid. iv. 38. 62, ed. Hultsch 274. 18–276. 14

λη′. Δεδειγμένου δὴ τούτου τρίχα τέμνεται ἡ δοθεῖσα γωνία εὐθύγραμμος οὕτως.

Ἔστω γὰρ ὀξεῖα πρότερον ἡ ὑπὸ ΑΒΓ, καὶ ἀπό τινος σημείου κάθετος ἡ ΑΓ, καὶ συμπληρωθέντος τοῦ ΓΖ παραλληλογράμμου ἡ ΖΑ ἐκβεβλήσθω ἐπὶ



τὸ Ε, καὶ παραλληλογράμμου ὄντος ὀρθογωνίου τοῦ ΓΖ κείσθω μεταξὺ τῶν ΕΑΓ εὐθεῖα ἡ ΕΔ νεύουσα ἐπὶ τὸ Β ἴση τῇ διπλασίᾳ τῆς ΑΒ (τοῦτο γὰρ ὡς δυνατὸν γενέσθαι προγέγραπται)· λέγω δὴ ὅτι τῆς δοθείσης γωνίας τῆς ὑπὸ ΑΒΓ τρίτον μέρος ἐστὶν ἡ ὑπὸ ΕΒΓ.

to EZ, EΔ. Since ZE is given and is equal to ΔH, therefore ΔH is also given. And Δ is given; therefore H is on the circumference of a circle given in position. And since the rectangle contained by BΓ, ΓΔ is given and is equal to the rectangle contained by BZ, EΔ [Eucl. i. 43], therefore the rectangle contained by BZ, EΔ is given, that is, the rectangle contained by BZ, ZH is given; therefore H lies on a hyperbola. But it is also on the circumference of a circle given in position; therefore H is given.[a]

Ibid. iv. 38. 62, ed. Hultsch. 274. 18–276. 14

38. With this proved, the given rectilineal angle is trisected in the following manner.

First let ABΓ be an acute angle, and from any point [of the straight line AB] let the perpendicular AΓ be drawn, and let the parallelogram ΓZ be completed, and let ZA be produced to E, and inasmuch as ΓZ is a right-angled parallelogram let the straight line EΔ be placed between EA, AΓ so as to verge towards B and be equal to twice AB—that this is possible has been proved above; I say that EBΓ is a third part of the given angle ABΓ.

[a] The formal synthesis then follows as Pappus iv. 37.

Τετμήσθω γὰρ ἡ ΕΔ δίχα τῷ Η, καὶ ἐπεζεύχθω ἡ ΑΗ· αἱ τρεῖς ἄρα αἱ ΔΗ, ΗΑ, ΗΕ ἴσαι εἰσίν· διπλῆ ἄρα ἡ ΔΕ τῆς ΑΗ. ἀλλὰ καὶ τῆς ΑΒ διπλῆ· ἴση ἄρα ἐστὶν ἡ ΒΑ τῇ ΑΗ, καὶ ἡ ὑπὸ ΑΒΔ γωνία τῇ ὑπὸ ΑΗΔ. ἡ δὲ ὑπὸ ΑΗΔ διπλασία τῆς ὑπὸ ΑΕΔ, τουτέστιν τῆς ὑπὸ ΔΒΓ· καὶ ἡ ὑπὸ ΑΒΔ ἄρα διπλῆ ἐστιν τῆς ὑπὸ ΔΒΓ. καὶ ἐὰν τὴν ὑπὸ ΑΒΔ δίχα τέμωμεν, ἔσται ἡ ὑπὸ ΑΒΓ γωνία τρίχα τετμημένη.

(c) Direct Solutions by Means of Conics

Ibid. iv. 43. 67–44. 68, ed. Hultsch 280. 20–284. 20

μγ΄. Καὶ ἄλλως τῆς δοθείσης περιφερείας τὸ

---

[a] We may easily show with Heath (*H.G.M.* i. 237-238) how the solution of the νεῦσις is equivalent to the solution of a cubic equation. If in the accompanying figure ΖΕ, ΖΒ are the axes of x, y respectively, and $ΖΑ = a$, $ΖΒ = b$, the point Θ giving Ε is determined as the intersection of the circle

$$(x-a)^2+(y-b)^2=4(a^2+b^2)$$

and the hyperbola $$xy=ab.$$

By eliminating x from these equations we may obtain

$$(y+b)(y^3-3by^2-3a^2y+a^2b)=0.$$

One of the points of intersection of the circle and hyperbola is therefore given by $y=-b$, $x=-a$.

The other three are determined by the equation

$$y^3-3by^2-3a^2y+a^2b=0.$$

For let EΔ be bisected at H, and let AH be joined; the three straight lines ΔH, HA, HE are therefore equal; therefore ΔE is double of AH. But it is also double of AB; therefore BA is equal to AH, and the angle ABΔ is equal to AHΔ. Now AHΔ is double of AEΔ, that is, of ΔBΓ; and therefore ABΔ is double of ΔBΓ. And if we bisect ABΔ, the angle ABΓ will be trisected.[a]

(c) Direct Solutions by Means of Conics

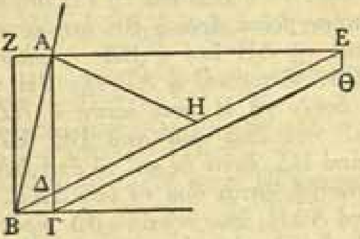
Ibid. iv. 43. 67–44. 68, ed. Hultsch 280. 20–284. 20

43. Another way of cutting off the third part of a

If $\qquad \angle AB\Gamma = \theta$, so that $\tan\theta = \dfrac{b}{a}$,

and $\qquad \tau = \tan \Delta B\Gamma$, so that $y = a\tau$,

then $\qquad a^3\tau^3 - 3ba^2\tau^2 - 3a^3\tau + a^2b = 0$



i.e. $\qquad a\tau^3 - 3b\tau^2 - 3a\tau + b = 0$

whence $\qquad b(1 - 3\tau^2) = a(3\tau - \tau^3)$

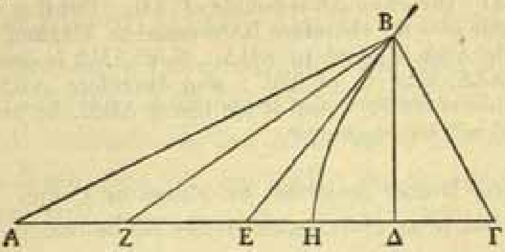
and $\qquad \tan\theta = \dfrac{b}{a} = \dfrac{3\tau - \tau^3}{1 - 3\tau^2}$.

Accordingly, by a well-known theorem in trigonometry,

$$\tau = \tan \tfrac{1}{3}\theta,$$

and $\angle AB\Gamma$ is trisected by EB.

τρίτον ἀφαιρεῖται μέρος, χωρὶς τῆς νεύσεως, διὰ στερεοῦ τόπου τοιούτου.



Θέσει ἡ διὰ τῶν Α, Γ, *καὶ ἀπὸ δοθέντων ἐπ' αὐτῆς τῶν* Α, Γ *κεκλάσθω ἡ* ΑΒΓ *διπλασίαν ποιοῦσα τὴν ὑπὸ* ΑΓΒ *γωνίαν τῆς ὑπὸ* ΓΑΒ· *ὅτι τὸ* Β *πρὸς ὑπερβολῇ.*

Ἤχθω κάθετος ἡ ΒΔ, *καὶ τῇ* ΓΔ *ἴση ἀπειλήφθω ἡ* ΔΕ· *ἐπιζευχθεῖσα ἄρα ἡ* ΒΕ *ἴση ἔσται τῇ* ΑΕ. *κείσθω καὶ τῇ* ΔΕ *ἴση ἡ* ΕΖ· *τριπλασία ἄρα ἡ* ΓΖ *τῆς* ΓΔ. *ἔστω καὶ ἡ* ΑΓ *τῆς* ΓΗ *τριπλασία· ἔσται δὴ δοθὲν τὸ* Η, *καὶ λοιπὴ ἡ* ΑΖ *τῆς* ΗΔ *τριπλασία. καὶ ἐπεὶ τῶν ἀπὸ* ΒΕ, ΕΖ *ὑπεροχή ἐστιν τὸ ἀπὸ* ΒΔ, *ἔστιν δὲ καὶ τὸ ὑπὸ* ΔΑ, ΑΖ *τῶν αὐτῶν ὑπεροχή, ἔσται ἄρα τὸ ὑπὸ* ΔΑΖ, *τουτέστιν τὸ τρὶς ὑπὸ* ΑΔΗ, *ἴσον τῷ ἀπὸ* ΒΔ· *πρὸς ὑπερβολῇ ἄρα τὸ* Β, *ἧς πλαγία μὲν τοῦ πρὸς ἄξονι εἴδους ἡ*

---

[a] For by the equality of the triangles ΒΕΔ, ΒΓΔ, we have ∠ΒΕΓ = ∠ΒΓΕ = 2∠ΓΑΒ (*ex hypothesi*). But ∠ΒΕΓ = ∠ΓΑΒ + ∠ΑΒΕ.

Therefore ∠ΓΑΒ = ∠ΑΒΕ, and so ΒΕ = ΑΕ.

[b] *i.e.* since ΓΗ = $\frac{1}{3}$ΑΓ and ΓΔ = $\frac{1}{3}$ΓΖ, by subtraction,

ΓΗ − ΓΔ = $\frac{1}{3}$(ΑΓ − ΓΖ), or ΗΔ = $\frac{1}{3}$ΑΖ.

given arc is furnished, without the use of a verging, by this solid locus.

Let the straight line through Α, Γ be given in position, and from the given points Α, Γ upon it let ΑΒΓ be inflected, making the angle ΑΓΒ double of ΓΑΒ; I say that Β lies on a hyperbola.

For let ΒΔ be drawn perpendicular [to ΑΓ] and let ΔΕ be cut off equal to ΓΔ; when ΒΕ is joined it will therefore be equal to ΑΕ.[a] And let ΕΖ be placed equal to ΔΕ; therefore ΓΖ = 3ΓΔ. Now let ΓΗ be placed equal to $\frac{1}{3}$ΑΓ; therefore the point Η will be given, and the remainder[b] ΑΖ will equal 3ΗΔ.

Now since[c] $\quad \mathrm{BE}^2 - \mathrm{EZ}^2 = \mathrm{B}\Delta^2$,

and $\quad \mathrm{BE}^2 - \mathrm{EZ}^2 = \Delta\mathrm{A} \cdot \mathrm{AZ}$,

therefore $\quad \Delta\mathrm{A} \cdot \mathrm{AZ} = \mathrm{B}\Delta^2$,

that is $\quad 3\mathrm{A}\Delta \cdot \Delta\mathrm{H} = \mathrm{B}\Delta^2$;

therefore Β lies on a hyperbola with transverse axis

[a] The reasoning here is much abbreviated, and in full may be written as follows:

$$\begin{aligned} \mathrm{BE}^2 - \mathrm{EZ}^2 &= \mathrm{BE}^2 - \mathrm{E}\Delta^2 \text{ (since EZ = E}\Delta\textit{ ex hypothesi}) \\ &= \mathrm{B}\Delta^2 \qquad \text{(Eucl. i. 47)} \end{aligned}$$

$$\begin{aligned} \text{Now } \mathrm{BE}^2 - \mathrm{EZ}^2 &= \mathrm{AE}^2 - \mathrm{EZ}^2 \text{ (since BE was proved equal to AE)} \\ &= \Delta\mathrm{A} \cdot \mathrm{AZ} \quad \text{(Eucl. ii. 6)} \end{aligned}$$

$\therefore \quad \Delta\mathrm{A} \cdot \mathrm{AZ} = \mathrm{B}\Delta^2$

$\therefore \quad 3\mathrm{A}\Delta \cdot \Delta\mathrm{H} = \mathrm{B}\Delta^2$ (since ΑΖ was proved equal to 3ΗΔ)

$$\therefore \mathrm{B}\Delta^2 : \mathrm{A}\Delta \cdot \Delta\mathrm{H} = 3 : 1 = \frac{3\mathrm{AH}^2}{\mathrm{AH}^2};$$

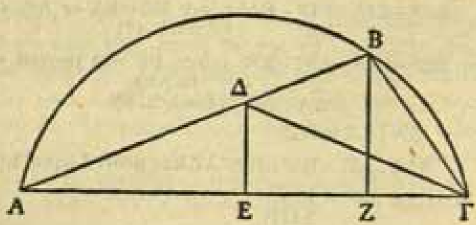
∴ Β lies on a hyperbola with transverse axis ΑΗ and conjugate axis $\sqrt{3}$ΑΗ.

ΑΗ, ἡ δὲ ὀρθία τριπλασία τῆς ΑΗ. καὶ φανερὸν ὅτι τὸ Γ σημεῖον ἀπολαμβάνει πρὸς τῇ Η κορυφῇ τῆς τομῆς τὴν ΓΗ ἡμίσειαν τῆς πλαγίας τοῦ εἴδους πλευρᾶς τῆς ΑΗ.

Καὶ ἡ σύνθεσις φανερά· δεήσει γὰρ τὴν ΑΓ τεμεῖν ὥστε διπλασίαν εἶναι τὴν ΑΗ τῆς ΗΓ, καὶ περὶ ἄξονα τὸν ΑΗ γράψαι διὰ τοῦ Η ὑπερβολήν, ἧς ὀρθία τοῦ εἴδους πλευρὰ τριπλασία τῆς ΑΗ, καὶ δεικνύναι ποιοῦσαν αὐτὴν τὸν εἰρημένον διπλάσιον λόγον τῶν γωνιῶν. καὶ ὅτι τῆς δοθείσης κύκλου περιφερείας τὸ γʹ ἀποτέμνει μέρος ἡ τοῦτον γραφομένη τὸν τρόπον ὑπερβολὴ συνιδεῖν ῥᾴδιον τῶν Α, Γ σημείων περάτων τῆς περιφερείας ὑποκειμένων.

μδʹ. Ἑτέρως δὲ τὴν ἀνάλυσιν τοῦ τρίχα τεμεῖν τὴν γωνίαν ἢ περιφέρειαν ἐξέθεντό τινες ἄνευ τῆς νεύσεως. ἔστω δὲ ἐπὶ περιφερείας ὁ λόγος· οὐδὲν γὰρ διαφέρει γωνίαν ἢ περιφέρειαν τεμεῖν.

Γεγονέτω δή, καὶ τῆς ΑΒΓ περιφερείας τρίτον



ἀπειλήφθω μέρος ἡ ΒΓ, καὶ ἐπεζεύχθωσαν αἱ ΑΒ, ΒΓ, ΓΑ· διπλασίων ἄρα ἡ ὑπὸ ΑΓΒ τῆς ὑπὸ

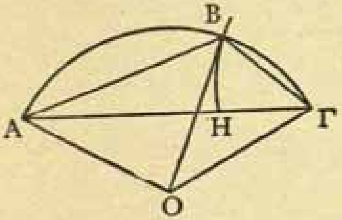
AH and conjugate axis $\sqrt{3}$AH. And it is clear that the point Γ cuts off at the vertex H of the [conic] section a straight line ΓH which is one-half of the transverse axis AH.

And the synthesis is clear; for it will be required so to cut AΓ that AH is double of HΓ, and about AH as axis to describe through H a hyperbola with conjugate axis $\sqrt{3}$AH, and to prove that it makes the aforementioned double ratio of the angles. And that the hyperbola described in this manner cuts off the third part of the arc of the given circle is easily understood if the points A, Γ are the end points of the arc.[a]

44. Some set out differently the analysis of the problem of trisecting an angle or arc without a verging. Let the ratio be upon an arc; it makes no difference whether an angle or an arc is to be divided.

Let it be done, and let BΓ, the third part of the arc ABΓ, be cut off, and let AB, BΓ, ΓA be joined; then

[a] For let O be the centre of a circle of which AΓ is an arc. Let AΓ be divided at H so that AH $=2$HΓ. Let the hyperbola be constructed which has AH for transverse axis and $\sqrt{3}$ AH for conjugate axis, and let this hyperbola cut the arc of the circle in B. Then by Pappus's proposition,



$$\angle B\Gamma A = 2\angle BA\Gamma.$$

Therefore their doubles are equal,

or $$\angle BOA = 2\angle BO\Gamma,$$

and so OB trisects the angle AOΓ and the arc AB.

ΒΑΓ. *τετμήσθω δίχα ἡ ὑπὸ* ΑΓΒ *τῇ* ΓΔ, *καὶ κάθετοι αἱ* ΔΕ, ΖΒ· *ἴση ἄρα ἡ* ΑΔ *τῇ* ΔΓ, *ὥστε καὶ ἡ* ΑΕ *τῇ* ΕΓ· *δοθὲν ἄρα τὸ* Ε. *ἐπεὶ οὖν ἐστιν ὡς ἡ* ΑΓ *πρὸς* ΓΒ, *οὕτως ἡ* ΑΔ *πρὸς* ΔΒ, *τουτέστιν ἡ* ΑΕ *πρὸς* ΕΖ, *καὶ ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ* ΓΑ *πρὸς* ΑΕ, *ἡ* ΒΓ *πρὸς* ΕΖ. *διπλῆ δὲ ἡ* ΓΑ *τῇ* ΑΕ· *διπλῆ ἄρα καὶ ἡ* ΒΓ *τῆς* ΕΖ· *τετραπλάσιον ἄρα τὸ ἀπὸ* ΒΓ, *τουτέστιν τὰ ἀπὸ τῶν* ΒΖΓ, *τοῦ ἀπὸ τῆς* ΕΖ. *ἐπεὶ οὖν δύο δοθέντα ἐστὶν τὰ* Ε, Γ, *καὶ ὀρθὴ ἡ* ΒΖ, *καὶ λόγος ἐστὶν τοῦ ἀπὸ* ΕΖ *πρὸς τὰ ἀπὸ τῶν* ΒΖΓ, *τὸ* Β *ἄρα πρὸς ὑπερβολῇ. ἀλλὰ καὶ πρὸς θέσει περιφερείᾳ· δοθὲν ἄρα τὸ* Β. *καὶ ἡ σύνθεσις φανερά.*

---

[a] The relation ΒΓ = 2ΕΖ tells us that Β lies on a hyperbola with foci Α, Γ, directrix ΒΖ and eccentricity 2. Pappus proceeds to turn this into the axial form $\text{ΕΖ}^2 : \text{ΒΖ}^2 + \text{ΖΓ}^2 = 1 : 4$ which was more commonly used by the Greeks. In fact, there are only two other extant passages in which the focus-directrix property is used. One of them is also given by Pappus (vii., ed. Hultsch 1004-1014), who there proved

$\angle A\Gamma B = 2\angle BA\Gamma$. Let $\angle A\Gamma B$ be bisected by $\Gamma\Delta$, and let ΔE, ZB be drawn perpendicular; therefore $A\Delta$ is equal to $\Delta\Gamma$, so that AE is also equal to $E\Gamma$; therefore E is given.

Now because $\quad A\Gamma : \Gamma B = A\Delta : \Delta B \quad$ [Eucl. vi. 5

$$= AE : EZ,$$

therefore alternately $\Gamma A : AE = B\Gamma : EZ$.
But $\Gamma A = 2AE$; and therefore $B\Gamma = 2EZ$; therefore $B\Gamma^2 = 4EZ^2$, that is, $BZ^2 + Z\Gamma^2 = 4EZ^2$. Now, since the two points E, Γ are given, and BZ is drawn at right angles, and the ratio $EZ^2 : BZ^2 + Z\Gamma^2$ is given, B lies on a hyperbola. But it also lies on an arc given in position; therefore B is given. And the synthesis is clear.[a]

generally that "if the distance of a point from a fixed point is in a given ratio to its distance from a fixed line, the locus of the point is a conic section which is an ellipse, a parabola or a hyperbola according as the given ratio is less than, equal to, or greater than, unity." The proof is among a number of lemmas to the *Surface Loci* of Euclid, so presumably the focus-directrix property was already well known when Euclid wrote.

X. ZENO OF ELEA

X. ZENO OF ELEA[a]

Aristot. *Phys.* Z 9, 239 b 5–240 a 18

Ζήνων δὲ παραλογίζεται· εἰ γὰρ ἀεί, φησίν, ἠρεμεῖ πᾶν ἢ κινεῖται[1] ὅταν ᾖ κατὰ τὸ ἴσον, ἔστιν δ' ἀεὶ τὸ φερόμενον ἐν τῷ νῦν, ἀκίνητον τὴν φερομένην εἶναι ὀιστόν. τοῦτο δ' ἐστὶ ψεῦδος· οὐ γὰρ σύγκειται ὁ χρόνος ἐκ τῶν νῦν τῶν ἀδιαιρέτων, ὥσπερ οὐδ' ἄλλο μέγεθος οὐδέν.

Τέτταρες δ' εἰσὶν οἱ λόγοι περὶ κινήσεως Ζήνωνος οἱ παρέχοντες τὰς δυσκολίας τοῖς λύουσιν, πρῶτος μὲν ὁ περὶ τοῦ μὴ κινεῖσθαι διὰ τὸ πρότερον εἰς τὸ ἥμισυ δεῖν ἀφικέσθαι τὸ φερόμενον ἢ πρὸς τὸ τέλος, περὶ οὗ διείλομεν ἐν τοῖς πρότερον λόγοις.

[1] Zeller would bracket ἢ *κινεῖται*, and he is followed by Ross, but not, it seems to me, with sufficient reason. Diels, followed by Lee, has the unnecessary addition of *οὐδὲν δὲ κινεῖται* after these words. The passage as it stands is satisfactorily explained by Brochard (*Études de philosophie ancienne et de philosophie moderne*, p. 6) and by Heath (*H.G.M.* i. 276).

[a] Zeno of Elea, who is represented by Plato (*Parm.* 127 B) as "about forty" when Socrates was a "very young man" (say in 450 B.C.), was a disciple of Parmenides. The object of his four arguments on motion, here reproduced from Aristotle, was to show that the rejection of Parmenides' doctrine of the unity of being led to self-contradictory results.

X. ZENO OF ELEA[a]

Aristotle, *Physics* Z 9, 239 b 5–240 a 18

ZENO's argument is fallacious; for, he says, if everything is either at rest or in motion when it occupies a space equal to itself, while the object moved is always in the instant, the moving arrow is unmoved. But this is false; for time is not made up of indivisible instants, any more than is any other magnitude.

Zeno has four arguments about motion which present difficulties to those who try to resolve them. The first is that which says there is no motion because the object moved must arrive at the middle before it arrives at the end,[b] concerning which we have already treated.

[a] A vast literature has grown round these arguments, but the student will find most help in W. D. Ross, *Aristotle's Physics*, pp. 655-666, H. D. P. Lee, *Zeno of Elea*, and Heath, *H.G.M.* i. 271-283.

[b] Not only has it to pass through the half-way point, but through half of the remaining half, and so on to infinity. If a is the length of the course measured from the goal, then the moving object before it reaches its goal has to pass through the points $\frac{a}{2}, \frac{a}{2^2}, \frac{a}{2^3} \ldots$ and so on through an infinite series which cannot be enumerated. Aristotle's answer is that the moving object has indeed to pass through an infinite number of positions, but in a finite time it has an infinite number of instants in which to do so.

Διὸ καὶ ὁ Ζήνωνος λόγος ψεῦδος λαμβάνει τὸ μὴ ἐνδέχεσθαι τὰ ἄπειρα διελθεῖν ἢ ἅψασθαι τῶν ἀπείρων καθ' ἕκαστον ἐν πεπερασμένῳ χρόνῳ. διχῶς γὰρ λέγεται καὶ τὸ μῆκος καὶ ὁ χρόνος ἄπειρον, καὶ ὅλως πᾶν τὸ συνεχές, ἤτοι κατὰ διαίρεσιν ἢ τοῖς ἐσχάτοις. τῶν μὲν οὖν κατὰ τὸ ποσὸν ἀπείρων οὐκ ἐνδέχεται ἅψασθαι ἐν πεπερασμένῳ χρόνῳ, τῶν δὲ κατὰ διαίρεσιν ἐνδέχεται· καὶ γὰρ αὐτὸς ὁ χρόνος οὕτως ἄπειρος. ὥστε ἐν τῷ ἀπείρῳ καὶ οὐκ ἐν τῷ πεπερασμένῳ συμβαίνει διιέναι τὸ ἄπειρον, καὶ ἅπτεσθαι τῶν ἀπείρων τοῖς ἀπείροις, οὐ τοῖς πεπερασμένοις.

Δεύτερος δ' ὁ καλούμενος Ἀχιλλεύς· ἔστι δ' οὗτος, ὅτι τὸ βραδύτατον οὐδέποτε καταληφθήσεται θέον ὑπὸ τοῦ ταχίστου· ἔμπροσθεν γὰρ ἀναγκαῖον ἐλθεῖν τὸ διῶκον, ὅθεν ὥρμησε τὸ φεῦγον, ὥστ' ἀεί τι προέχειν ἀναγκαῖον τὸ βραδύτερον. ἔστι δὲ καὶ οὗτος ὁ αὐτὸς λόγος τῷ διχοτομεῖν, διαφέρει δ' ἐν τῷ διαιρεῖν μὴ δίχα τὸ προσλαμβανόμενον μέγεθος. τὸ μὲν οὖν μὴ καταλαμβάνεσθαι τὸ βραδύτερον συμβέβηκεν ἐκ τοῦ λόγου, γίγνεται δὲ παρὰ ταὐτὸ τῇ διχοτομίᾳ (ἐν ἀμφοτέροις γὰρ συμβαίνει μὴ ἀφικνεῖσθαι πρὸς τὸ πέρας διαιρου-

---

[a] The passage between the asterisks, to which Aristotle refers the reader, is *Phys.* Z 2, 233 a 21-31 and is reproduced here for convenience.

[b] Aristotle's argument is correct. The *Achilles* is a more general form of the *Dichotomy*. If the speed of Achilles is n times that of the tortoise (we learn from Themistius and Simplicius that the tortoise was the object pursued), and the tortoise starts a unit ahead, then when Achilles has reached the point where the tortoise started the

* Zeno's argument makes a false assumption in not allowing the possibility of passing through or touching an infinite number of positions one by one in a limited time. For there are two senses in which length and time, and, generally, any continuum, are said to be infinite, either in respect of division or of extension. So where the infinite is infinite in respect of quantity, it is not possible to make in a limited time an infinite number of contacts, but it is possible where the infinite is infinite in respect of division; for the time also is infinite in this respect. And so it is possible to pass through an infinite number of positions in a time which is in this sense infinite, but not in a time which is finite, and to make an infinite number of contacts because its moments are infinite, not finite.* [a]

The second argument is the so-called *Achilles*; this asserts that the slowest will never be overtaken by the quickest; for that which is pursuing must first reach the point from which the fleeing object started, so that the slower must necessarily always be some distance ahead. This is the same reasoning as that of the *Dichotomy*, the only difference being that when the magnitude which is successively added is divided it is not necessarily bisected.[b] The argument leads to the conclusion that the slower will never be overtaken, and it is for the same reason as in the *Dichotomy* (for in both by dividing the distance in some way it is

tortoise is $\frac{1}{n}$ ahead; when Achilles has reached this point the tortoise is $\frac{1}{n^2}$ ahead; and so on to infinity. Putting $n=2$ we get the special conditions of the *Dichotomy*. Both arguments emphasize that to traverse a finite distance means passing through an infinite number of positions.

μένου πως τοῦ μεγέθους· ἀλλὰ πρόσκειται ἐν τούτῳ ὅτι οὐδὲ τὸ τάχιστον τετραγῳδημένον ἐν τῷ διώκειν τὸ βραδύτατον), ὥστ' ἀνάγκη καὶ τὴν λύσιν εἶναι τὴν αὐτήν. τὸ δ' ἀξιοῦν ὅτι τὸ προέχον οὐ καταλαμβάνεται, ψεῦδος· ὅτε γὰρ προέχει, οὐ καταλαμβάνεται· ἀλλ' ὅμως καταλαμβάνεται, εἴπερ δώσει διεξιέναι τὴν πεπερασμένην.

Οὗτοι μὲν οὖν οἱ δύο λόγοι, τρίτος δ' ὁ νῦν ῥηθείς, ὅτι ἡ ὀιστὸς φερομένη ἕστηκεν. συμβαίνει δὲ παρὰ τὸ λαμβάνειν τὸν χρόνον συγκεῖσθαι ἐκ τῶν νῦν· μὴ διδομένου γὰρ τούτου οὐκ ἔσται ὁ συλλογισμός.

Τέταρτος δ' ὁ περὶ τῶν ἐν τῷ σταδίῳ κινουμένων ἐξ ἐναντίας ἴσων ὄγκων παρ' ἴσους, τῶν μὲν ἀπὸ τέλους τοῦ σταδίου τῶν δ' ἀπὸ μέσου, ἴσῳ τάχει, ἐν ᾧ συμβαίνειν οἴεται ἴσον εἶναι χρόνον τῷ διπλασίῳ τὸν ἥμισυν. ἔστι δ' ὁ παραλογισμὸς

---

[a] Achilles overtakes the tortoise when he has travelled a distance

$$1+\frac{1}{n}+\frac{1}{n^2}+ \ . \ . \ . \ ad\ inf.$$

This is a convergent series whose sum is $\frac{n}{n-1}$. The ancients did not know how to sum an infinite series, but they knew that Achilles would catch the tortoise and that the problem *solvitur ambulando*.

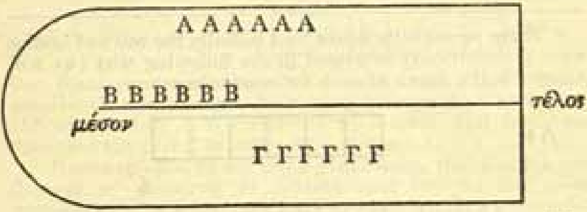
[b] Lachelier (*Revue de métaphysique et de morale*, xviii., pp. 346-347) and Ross explain that ἀπὸ τοῦ μέσου means from the turning point in the double course or δίαυλος. The race was from the τέλος to the μέσον and back again to the τέλος. On this interpretation it is possible to translate easily and naturally. Gaye, the Oxford translators and Lee, who do not accept this interpretation, but believe τὸ μέσον to refer

concluded that the goal will not be reached; but in this a dramatic effect is produced by saying that not even the swiftest will be successful in its pursuit of the slowest) and so the solution must necessarily be the same. The claim that the one in front is not overtaken is false; for when in front he is not indeed overtaken, but he will nevertheless be overtaken if he give his pursuer a finite distance to go through.[a]

These are two of the arguments, and the third is the one just mentioned, that the flying arrow is at rest. This conclusion follows from the assumption that time is composed of instants; for if this is not granted the reasoning does not follow.

The fourth is that about the two rows of equal bodies moving past each other in the stadium with equal velocities in opposite directions, the one row starting from the end of the stadium, the other from the middle.[b] This, he thinks, leads to the conclusion that half a given time is equal to its double. The

to the middle of the A s, are forced to paraphrase: "The

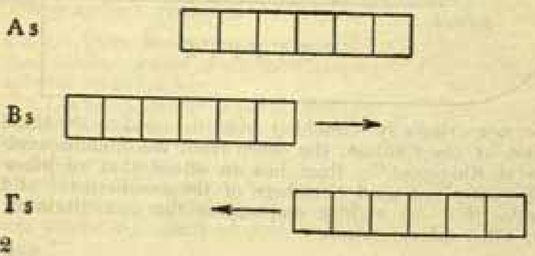


one row originally stretching from the goal to the middle-point of the stadium, the other from the middle-point to the starting-post." Ross has to admit that τὸ μέσον is apparently not used elsewhere of the middle-point of the δίαυλος, but he rightly emphasizes the unnaturalness of any other interpretation.

ἐν τῷ τὸ μὲν παρὰ κινούμενον τὸ δὲ παρ' ἠρεμοῦν τὸ ἴσον μέγεθος ἀξιοῦν τῷ ἴσῳ τάχει τὸν ἴσον φέρεσθαι χρόνον· τοῦτο δ' ἐστὶ ψεῦδος. οἷον ἔστωσαν οἱ ἑστῶτες ἴσοι ὄγκοι ἐφ' ὧν τὰ ΑΑ, οἱ δ' ἐφ' ὧν τὰ ΒΒ ἀρχόμενοι ἀπὸ τοῦ μέσου, ἴσοι τὸν ἀριθμὸν τούτοις ὄντες καὶ τὸ μέγεθος, οἱ δ' ἐφ' ὧν τὰ ΓΓ ἀπὸ τοῦ ἐσχάτου, ἴσοι τὸν ἀριθμὸν ὄντες τούτοις καὶ τὸ μέγεθος, καὶ ἰσοταχεῖς τοῖς Β. συμβαίνει δὴ τὸ πρῶτον Β ἅμα ἐπὶ τῷ ἐσχάτῳ εἶναι καὶ τὸ πρῶτον Γ, παρ' ἄλληλα κινουμένων. συμβαίνει δὲ τὸ Γ παρὰ πάντα [τὰ Β][1] διεξεληλυθέναι, τὸ δὲ Β παρὰ τὰ ἡμίση· ὥστε ἥμισυν εἶναι τὸν χρόνον· ἴσον γὰρ ἑκάτερόν ἐστι παρ'

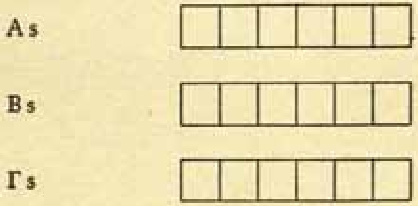
[1] τὰ Β del. Ross.

[a] There seems little doubt that initially the rows of bodies were symmetrically arranged in the following way (we will assume half a dozen of each for convenience):



fallacy lies in assuming that a body takes an equal time to pass with equal speed a body in motion and a body of equal size at rest; but this is untrue. For example, let AA be stationary bodies of equal size, let BB be the bodies equal in number and size that start from the middle, and let ΓΓ be the bodies equal in number and size that start from the end, having a speed equal to that of the B s.[a] In consequence, the first B and the first Γ move past each other and come simultaneously to the end.[b] It follows that Γ has passed all the bodies it is moving past, though B has passed only half the bodies it is moving past,[c] so that B has taken half the time [taken by Γ]; for

and that the final position they take up is:



But there are great difficulties in the text. Ross's interpretation seems to me to do least violence to the Greek.

[b] *i.e.* the first B is under the right-hand A at the same time that the first Γ is under the left-hand A.

[c] Ross explains, to my mind judiciously, that the B s are thought of primarily as moving past the A s and only secondarily as moving past the Γ s, while the Γ s are thought of primarily as moving past the B s and only secondarily past the A s. Zeno wishes to point out that the first B has moved past only three A s while the first Γ has moved past six B s. On the ground that to move past six B s requires twice the time needed to move past three A s, coupled with the knowledge that the time taken is in fact the same in

XI. THEAETETUS

XI. THEAETETUS

(a) General

Suidas, *s.v.* Θεαίτητος

Θεαίτητος, Ἀθηναῖος, ἀστρολόγος, φιλόσοφος, μαθητὴς Σωκράτους, ἐδίδαξεν ἐν Ἡρακλείᾳ. πρῶτος δὲ τὰ πέντε καλούμενα στερεὰ ἔγραψε. γέγονε δὲ μετὰ τὰ Πελοποννησιακά.

(b) The Five Regular Solids

Schol. i. in Eucl. *Elem.* xiii., Eucl. ed. Heiberg v. 654

Ἐν τούτῳ τῷ βιβλίῳ, τουτέστι τῷ ιγ', γράφεται τὰ λεγόμενα Πλάτωνος ε̄ σχήματα, ἃ αὐτοῦ μὲν οὐκ ἔστιν, τρία δὲ τῶν προειρημένων ε̄ σχημάτων τῶν Πυθαγορείων ἐστίν, ὅ τε κύβος καὶ ἡ πυραμὶς καὶ τὸ δωδεκάεδρον, Θεαιτήτου δὲ τό τε ὀκτάεδρον καὶ τὸ εἰκοσάεδρον. τὴν δὲ προσωνυμίαν ἔλαβεν Πλάτωνος διὰ τὸ μεμνῆσθαι αὐτὸν ἐν τῷ Τιμαίῳ περὶ αὐτῶν.

---

[a] Theaetetus lived about 415–369 B.C. He is the subject of a dissertation *De Theaeteto Atheniensi* by Eva Sachs (Berlin, 1914).

XI. THEAETETUS [a]

(a) General

Suidas, *s.v. Theaetetus*

Theaetetus, an Athenian, astronomer, philosopher, a pupil of Socrates, taught in Heraclea. He was the first to describe [b] the five solids so-called. He lived after the Peloponnesian wars.

(b) The Five Regular Solids

Euclid, *Elements* xiii., Scholium i., Eucl. ed. Heiberg v. 654

In this book, that is, the thirteenth, are described the five Platonic figures, which are however not his, three of the aforesaid five figures being due to the Pythagoreans,[c] namely, the cube, the pyramid and the dodecahedron, while the octahedron and icosahedron are due to Theaetetus. They received the name Platonic because he discourses in the *Timaeus* about them.

[b] Possibly "construct."

[c] For the relation of the Pythagoreans to the five regular solids, see *supra*, pp. 216-225. Theaetetus was probably the first to construct all five theoretically: the Pythagoreans could not have done that. For a full discussion, see Eva Sachs, *Die fünf Platonischen Körper*.

(c) The Irrational

Schol. lxii. in Eucl. *Elem.* x., Eucl. ed. Heiberg v. 450. 16-18

Τὸ θεώρημα τοῦτο Θεαιτήτειόν ἐστιν εὕρημα, καὶ μέμνηται αὐτοῦ ὁ Πλάτων ἐν Θεαιτήτῳ, ἀλλ' ἐκεῖ μὲν μερικώτερον ἔγκειται, ἐνταῦθα δὲ καθόλου.

Plat. *Theaet.* 147 d–148 b

ΘΕΑΙΤΗΤΟΣ. Περὶ δυνάμεών τι ἡμῖν Θεόδωρος ὅδε ἔγραφε, τῆς τε τρίποδος πέρι καὶ πεντέποδος [ἀποφαίνων][1] ὅτι μήκει οὐ σύμμετροι τῇ ποδιαίᾳ, καὶ οὕτω κατὰ μίαν ἑκάστην προαιρούμενος μέχρι τῆς ἑπτακαιδεκάποδος· ἐν δὲ ταύτῃ πως ἐνέσχετο. ἡμῖν οὖν εἰσῆλθέ τι τοιοῦτον, ἐπειδὴ ἄπειροι τὸ πλῆθος αἱ δυνάμεις ἐφαίνοντο, πειραθῆναι συλλαβεῖν εἰς ἕν, ὅτῳ πάσας ταύτας προσαγορεύσομεν τὰς δυνάμεις.

[1] ἀποφαίνων secl. Burnet.

---

[a] The enunciation is: *The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number; and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length. But the squares on straight lines incommensurable in length have not to one another the ratio which a square number has to a square number; and squares which have not to one another the ratio which a square number has to a square number will not have their sides commensurable in length either.*

[b] Theodorus of Cyrene, claimed by Iamblichus (*Vit. Pythag.* 36) as a Pythagorean and said to have been Plato's teacher in mathematics (Diog. Laert. ii. 103).

[c] Several conjectures have been put forward to explain

(c) The Irrational

Euclid, *Elements* x., Scholium lxii., ed. Heiberg v. 450. 16-18

This theorem [Eucl. *Elem.* x. 9][a] is the discovery of Theaetetus, and Plato recalls it in the *Theaetetus*, but there it arises in a particular case, here it is treated generally.

Plato, *Theaetetus* 147 d–148 b

Theaetetus. Theodorus[b] was proving to us a certain thing about square roots, I mean the square roots of three square feet and five square feet, namely, that these roots are not commensurable in length with the foot-length, and he proceeded in this way, taking each case in turn up to the root of seventeen square feet ; at this point for some reason he stopped.[c] Now it occurred to us, since the number of square roots appeared to be unlimited, to try to gather them into one class, by which we could henceforth describe all the roots.

how Theodorus proved that $\sqrt{3}$, $\sqrt{5}$ . . . $\sqrt{17}$ are incommensurable. They are summarized by Heath (*H.G.M.* i. 204-208). One theory is that Theodorus adapted the traditional proof (*supra*, p. 110) of the incommensurability of $\sqrt{2}$. Another, put forward by Zeuthen (" Sur la constitution des livres arithmétiques des Éléments d'Euclide et leur rapport à la question de l'irrationalité " in *Oversigt over det kgl. Danske videnskabernes Selskabs Forhandlinger*, 1915, pp. 422 ff.), depends on the process of finding the greatest common measure as stated in Eucl. x. 2. If two magnitudes are such that the process of finding their G.C.M. never comes to an end, the two magnitudes are incommensurable. The method is simple in theory, but the geometrical application is fairly complicated, though doubtless not beyond the capabilities of Theodorus.

ΣΩΚΡΑΤΗΣ. Ἦ καὶ ηὕρετέ τι τοιοῦτον;

ΘΕΑΙ. Ἔμοιγε δοκοῦμεν· σκόπει δὲ καὶ σύ.

ΣΩ. Λέγε.

ΘΕΑΙ. Τὸν ἀριθμὸν πάντα δίχα διελάβομεν· τὸν μὲν δυνάμενον ἴσον ἰσάκις γίγνεσθαι τῷ τετραγώνῳ τὸ σχῆμα ἀπεικάσαντες τετράγωνόν τε καὶ ἰσόπλευρον προσείπομεν.

ΣΩ. Καὶ εὖ γε.

ΘΕΑΙ. Τὸν τοίνυν μεταξὺ τούτου, ὧν καὶ τὰ τρία καὶ τὰ πέντε καὶ πᾶς ὃς ἀδύνατος ἴσος ἰσάκις γενέσθαι, ἀλλ' ἢ πλείων ἐλαττονάκις ἢ ἐλάττων πλεονάκις γίγνεται, μείζων δὲ καὶ ἐλάττων ἀεὶ πλευρὰ αὐτὸν περιλαμβάνει, τῷ προμήκει αὖ σχήματι ἀπεικάσαντες προμήκη ἀριθμὸν ἐκαλέσαμεν.

ΣΩ. Κάλλιστα. ἀλλὰ τί τὸ μετὰ τοῦτο;

ΘΕΑΙ. Ὅσαι μὲν γραμμαὶ τὸν ἰσόπλευρον καὶ ἐπίπεδον ἀριθμὸν τετραγωνίζουσι, μῆκος ὡρισάμεθα, ὅσαι δὲ τὸν ἑτερομήκη, δυνάμεις, ὡς μήκει μὲν οὐ συμμέτρους ἐκείναις, τοῖς δ' ἐπιπέδοις ἃ δύνανται. καὶ περὶ τὰ στερεὰ ἄλλο τοιοῦτον.

---

[a] It is not possible to give the full force of the Greek as δυνάμεις, which literally means "powers," has to be trans-

Socrates. And did you find such a class?

Theaet. I think we did; but see if you agree.

Soc. Speak on.

Theaet. We divided all numbers into two classes. The one, consisting of numbers which can be represented as the product of equal factors, we likened in shape to the square and called them square and equilateral numbers.

Soc. And properly so.

Theaet. The numbers between these, among which are three and five and all that cannot be represented as the product of equal factors, but only as the product of a greater by a less or a less by a greater, and are therefore contained by greater and less sides, we likened to oblong shape and called oblong numbers.

Soc. Excellent. And what after this?

Theaet. Such lines as form the sides of equilateral plane numbers we called lengths, and such as form the oblong numbers we called roots, because they are not commensurable with the others in length, but only with the plane areas which they have the power to form.[a] And similarly in the case of solids.

lated "roots" to conform with mathematical usage. δυνάμεις, it will be noticed, are here limited to the square roots of oblong numbers, and are therefore always incommensurable.

SOCRATES. And did you find such a definition?

THEAETETUS. I think we did; but see if you agree.

SOC. Speak on.

THEAET. We divided all number into two classes. The one, consisting of numbers which can be represented as the product of equal factors, we likened in shape to the square, and we called them square and equilateral numbers.

SOC. And properly so.

THEAET. The numbers between these, among which are three and five and every number which cannot be represented as the product of equal factors, but only of a greater and a less or a less and a greater, and is therefore always contained by greater and less sides, we likened to the oblong figure and called oblong numbers.

SOC. Excellent. And what after this?

THEAET. Such lines as form the four sides of the equilateral plane numbers we called lengths, and those which form the oblong numbers we called roots, because they are not commensurable with the others in length, but only with the plane areas which they have the power to form. And similarly in the case of solids.

[illegible]

XII. PLATO

XII. PLATO

(a) General

Tzetzes, *Chil.* viii. 972-973

Πρὸ τῶν προθύρων τῶν αὑτοῦ γράψας ὑπῆρχε Πλάτων·

"Μηδεὶς ἀγεωμέτρητος εἰσίτω μου τὴν στέγην."

Plut. *Quaes. Conv.* viii. 2. 1

Ἐκ δὲ τούτου γενομένης σιωπῆς, πάλιν ὁ Διογενιανὸς ἀρξάμενος "βούλεσθ'," εἶπεν, "ἐπεὶ λόγοι περὶ θεῶν γεγόνασιν, ἐν τοῖς Πλάτωνος γενεθλίοις αὐτὸν Πλάτωνα κοινωνὸν παραλάβωμεν, ἐπισκεψάμενοι τίνα λαβὼν γνώμην ἀπεφήνατ' ἀεὶ γεωμετρεῖν τὸν θεόν ; εἴ γε δὴ θετέον εἶναι τὴν ἀπόφανσιν ταύτην Πλάτωνος." ἐμοῦ δὲ ταῦτ' εἰπόντος ὡς γέγραπται μὲν ἐν οὐδενὶ σαφῶς τῶν ἐκείνου βιβλίων, ἔχει δὲ πίστιν ἱκανὴν καὶ τοῦ Πλατωνικοῦ χαρακτῆρός ἐστιν.

Εὐθὺς ὑπολαβὼν ὁ Τυνδάρης "οἴει γάρ," εἶπεν, "ὦ Διογενιανέ, τῶν περιττῶν τι καὶ δυσθεωρήτων αἰνίττεσθαι τὸν λόγον, οὐχ ὅπερ αὐτὸς εἴρηκε καὶ γέγραφε πολλάκις, ὑμνῶν γεωμετρίαν, ὡς ἀπο-

---

[a] For Proclus's notice of Plato, see *supra*, p. 150, and for

XII. PLATO [a]

(a) General

Tzetzes, *Book of Histories* viii. 972-973

Over his front doors Plato wrote: "Let no one unversed in geometry come under my roof." [b]

Plutarch, *Convivial Questions* viii. 2. 1

Diogenianus broke the silence which followed this discussion by saying: "Since our discourse is about the gods, shall we make Plato share in it, especially as it is his birthday, and inquire what he meant when he said that God is for ever playing the geometer—if this saying is really Plato's?" I said that this saying is not plainly written in any of his works, but it is a credible saying and is of a Platonic character.

Thereupon Tyndares took up the discussion and said: "Do you think, Diogenianus, that this saying implies some subtle and recondite speculations, and not what he has so often mentioned, when he praises

the pseudo-Platonic instrument for finding two mean proportionals, *supra*, pp. 262-267. The mathematics in Plato is the subject of dissertations by C. Blass (*De Platone mathematico*, Bonn, 1861) and Seth Demel (*Platons Verhältnis zur Mathematik*, Leipzig, 1929).

[b] Johannes Tzetzes, the Byzantine pedant who lived in the twelfth century A.D., is not the best of authorities, so this charming story must be accepted with caution. The doors are presumably those of the Academy.

σπῶσαν ἡμᾶς προσισχομένους τῇ αἰσθήσει καὶ ἀποστρέφουσαν ἐπὶ τὴν νοητὴν καὶ ἀΐδιον φύσιν, ἧς θέα τέλος ἐστὶ φιλοσοφίας οἷον ἐποπτεία τελετῆς; . . . διὸ καὶ Πλάτων αὐτὸς ἐμέμψατο τοὺς περὶ Εὔδοξον καὶ Ἀρχύταν καὶ Μέναιχμον εἰς ὀργανικὰς καὶ μηχανικὰς κατασκευὰς τὸν τοῦ στερεοῦ διπλασιασμὸν ἀπάγειν ἐπιχειροῦντας, ὥσπερ πειρωμένους δι' ἀλόγου δύο μέσας ἀνάλογον, ᾗ παρείκοι, λαβεῖν· ἀπόλλυσθαι γὰρ οὕτω καὶ διαφθείρεσθαι τὸ γεωμετρίας ἀγαθὸν αὖθις ἐπὶ τὰ αἰσθητὰ παλινδρομούσης καὶ μὴ φερομένης ἄνω μηδ' ἀντιλαμβανομένης τῶν ἀιδίων καὶ ἀσωμάτων εἰκόνων, πρὸς αἷσπερ ὢν ὁ θεὸς ἀεὶ θεός ἐστι."

Aristox. *Harm.* ii. *ad. init.*, ed. Macran 122. 3-16

Βέλτιον ἴσως ἐστὶ τὸ προδιελθεῖν τὸν τρόπον τῆς πραγματείας τίς ποτ' ἐστίν, ἵνα προγιγνώσκοντες ὥσπερ ὁδὸν ᾗ βαδιστέον ῥᾴδιον πορευώμεθα εἰδότες τε κατὰ τί μέρος ἐσμὲν αὐτῆς καὶ μὴ λάθωμεν ἡμᾶς αὐτοὺς παρυπολαμβάνοντες τὸ πρᾶγμα. καθάπερ Ἀριστοτέλης ἀεὶ διηγεῖτο τοὺς πλείστους τῶν ἀκουσάντων παρὰ Πλάτωνος τὴν περὶ τἀγαθοῦ ἀκρόασιν παθεῖν· προσιέναι μὲν γὰρ ἕκαστον ὑπολαμβάνοντα λήψεσθαί τι τῶν νομιζομένων τούτων ἀνθρωπίνων ἀγαθῶν οἷον πλοῦτον ὑγίειαν ἰσχὺν τὸ ὅλον εὐδαιμονίαν τινὰ θαυμαστήν· ὅτε δὲ φανείησαν οἱ λόγοι περὶ μαθημάτων καὶ ἀριθμῶν καὶ γεωμετρίας καὶ ἀστρολογίας καὶ τὸ πέρας ὅτι ἀγαθόν ἐστιν ἕν, παντελῶς οἶμαι παράδοξόν τι

[a] The play on the words ἀλόγου, ἀνάλογον cannot be reproduced in English, but we may compensate ourselves by playing on the words "means," "mean proportionals."

geometry as a science that takes men away from sensible objects and turns them towards the intelligible and eternal, whose contemplation is the end of philosophy like the final grade of initiation into the mysteries? . . . Therefore Plato himself censured Eudoxus and Archytas and Menaechmus for endeavouring to solve the doubling of the cube by instruments and mechanical constructions, thus trying by irrational means to find two mean proportionals,[a] so far as that is allowable: for in this way what is good in geometry would be corrupted and destroyed, falling back again into sensible objects and not rising upwards and laying hold of immaterial and eternal images, among which God has his being and remains for ever God."

Aristoxenus, *Elements of Harmony* ii. *ad init.*, ed. Macran 122. 3-16

It is perhaps well to go through in advance the nature of our inquiry, so that, knowing beforehand the road along which we have to travel, we may have an easier journey, because we will know at what stage we are in, nor shall we harbour to ourselves a false conception of our subject. Such was the condition, as Aristotle often used to tell, of most of the audience who attended Plato's lecture on the Good. Every one went there expecting that he would be put in the way of getting one or other of the things accounted good in human life, such as riches or health or strength or, in fine, any extraordinary gift of fortune. But when they found that Plato's arguments were of mathematics and numbers and geometry and astronomy and that in the end he declared the One to be the Good, they were altogether taken by

ἐφαίνετο αὐτοῖς· εἶθ' οἱ μὲν ὑποκατεφρόνουν τοῦ πράγματος οἱ δὲ κατεμέμφοντο.

(b) Philosophy of Mathematics

Plat. *Rep.* vi. 510 c–e

Οἶμαι γάρ σε εἰδέναι ὅτι οἱ περὶ τὰς γεωμετρίας τε καὶ λογισμοὺς καὶ τὰ τοιαῦτα πραγματευόμενοι, ὑποθέμενοι τό τε περιττὸν καὶ τὸ ἄρτιον καὶ τὰ σχήματα καὶ γωνιῶν τριττὰ εἴδη καὶ ἄλλα τούτων ἀδελφὰ καθ' ἑκάστην μέθοδον, ταῦτα μὲν ὡς εἰδότες, ποιησάμενοι ὑποθέσεις αὐτά, οὐδένα λόγον οὔτε αὑτοῖς οὔτε ἄλλοις ἔτι ἀξιοῦσι περὶ αὐτῶν διδόναι ὡς παντὶ φανερῶν, ἐκ τούτων δ' ἀρχόμενοι τὰ λοιπὰ ἤδη διεξιόντες τελευτῶσιν ὁμολογουμένως ἐπὶ τοῦτο οὗ ἂν ἐπὶ σκέψιν ὁρμήσωσι.

Πάνυ μὲν οὖν, ἔφη, τοῦτό γε οἶδα.

Οὐκοῦν καὶ ὅτι τοῖς ὁρωμένοις εἴδεσι προσχρῶνται καὶ τοὺς λόγους περὶ αὐτῶν ποιοῦνται, οὐ περὶ τούτων διανοούμενοι, ἀλλ' ἐκείνων περὶ οἷς ταῦτα ἔοικε, τοῦ τετραγώνου αὐτοῦ ἕνεκα τοὺς λόγους ποιούμενοι καὶ διαμέτρου αὐτῆς, ἀλλ' οὐ ταύτης ἣν γράφουσιν, καὶ τἆλλα οὕτως, αὐτὰ μὲν ταῦτα ἃ πλάττουσίν τε καὶ γράφουσιν, ὧν καὶ σκιαὶ καὶ ἐν ὕδασιν εἰκόνες εἰσίν, τούτοις μὲν ὡς εἰκόσιν αὖ χρώμενοι, ζητοῦντες δὲ αὐτὰ ἐκεῖνα ἰδεῖν ἃ οὐκ ἂν ἄλλως ἴδοι τις ἢ τῇ διανοίᾳ.

Plat. *Ep.* vii. 342 a–343 b

Ἔστιν τῶν ὄντων ἑκάστῳ, δι' ὧν τὴν ἐπιστήμην ἀνάγκη παραγίγνεσθαι, τρία, τέταρτον δ' αὐτή—

surprise. The result was that some of them scoffed at the thing, while others found great fault with it.

(*b*) Philosophy of Mathematics

Plato, *Republic* vi. 510 c–e

I think you know that those who deal with geometries and calculations and such matters take for granted the odd and the even, figures, three kinds of angles and other things cognate to these in each field of inquiry; assuming these things to be known, they make them hypotheses, and henceforth regard it as unnecessary to give any explanation of them either to themselves or to others, treating them as if they were manifest to all; setting out from these hypotheses, they go at once through the remainder of the argument until they arrive with perfect consistency at the goal to which their inquiry was directed.

Yes, he said, I am aware of that.

Therefore I think you also know that although they use visible figures and argue about them, they are not thinking about these figures but of those things which the figures represent; thus it is the square in itself and the diameter in itself which are the matter of their arguments, not that which they draw; similarly, when they model or draw objects, which may themselves have images in shadows or in water, they use them in turn as images, endeavouring to see those absolute objects which cannot be seen otherwise than by thought.

Plato, *Epistle* vii. 342 a–343 b

For everything that exists there are three things through which knowledge about it must come; the

πέμπτον δ' αὐτὸ τιθέναι δεῖ ὃ δὴ γνωστόν τε καὶ ἀληθῶς ἐστιν ὄν—ἓν μὲν ὄνομα, δεύτερον δὲ λόγος, τὸ δὲ τρίτον εἴδωλον, τέταρτον δὲ ἐπιστήμη. περὶ ἓν οὖν λαβὲ βουλόμενος μαθεῖν τὸ νῦν λεγόμενον, καὶ πάντων οὕτω περὶ νόησον. κύκλος ἐστίν τι λεγόμενον, ᾧ τοῦτ' αὐτό ἐστιν ὄνομα ὃ νῦν ἐφθέγμεθα. λόγος δ' αὐτοῦ τὸ δεύτερον, ἐξ ὀνομάτων καὶ ῥημάτων συγκείμενος· τὸ γὰρ ἐκ τῶν ἐσχάτων ἐπὶ τὸ μέσον ἴσον ἀπέχον πάντῃ, λόγος ἂν εἴη ἐκείνου ᾧπερ στρογγύλον καὶ περιφερὲς ὄνομα καὶ κύκλος. τρίτον δὲ τὸ ζωγραφούμενόν τε καὶ ἐξαλειφόμενον καὶ τορνευόμενον καὶ ἀπολλύμενον· ὧν αὐτὸς ὁ κύκλος, ὃν περὶ πάντ' ἐστὶν ταῦτα, οὐδὲν πάσχει, τούτων ὡς ἕτερον ὄν. τέταρτον δὲ ἐπιστήμη καὶ νοῦς ἀληθής τε δόξα περὶ ταῦτ' ἐστίν· ὡς δὲ ἓν τοῦτο αὖ πᾶν θετέον, οὐκ ἐν φωναῖς οὐδ' ἐν σωμάτων σχήμασιν ἀλλ' ἐν ψυχαῖς ἐνόν, ᾧ δῆλον ἕτερόν τε ὂν αὐτοῦ τοῦ κύκλου τῆς φύσεως τῶν τε ἔμπροσθεν λεχθέντων τριῶν. τούτων δὲ ἐγγύτατα μὲν συγγενείᾳ καὶ ὁμοιότητι τοῦ πέμπτου νοῦς πεπλησίακεν, τἆλλα δὲ πλέον ἀπέχει. . . . κύκλος ἕκαστος τῶν ἐν ταῖς πράξεσι γραφομένων ἢ καὶ τορνευθέντων μεστὸς τοῦ ἐναντίου ἐστὶν τῷ πέμπτῳ—τοῦ γὰρ εὐθέος ἐφάπτεται πάντῃ—αὐτὸς δέ, φαμέν, ὁ κύκλος οὔτε τι σμικρότερον οὔτε μεῖζον τῆς ἐναντίας ἔχει ἐν αὑτῷ φύσεως. ὄνομά τε αὐτῶν φαμεν οὐδὲν οὐδενὶ βέβαιον εἶναι, κωλύειν δ' οὐδὲν τὰ νῦν στρογγύλα καλούμενα εὐθέα κεκλῆσθαι τά τε εὐθέα δὴ στρογγύλα, καὶ

knowledge itself is a fourth ; and as a fifth we must posit the actual object of knowledge which is the true reality. We have, then :—first, a name ; second, a description ; third, an image ; fourth, knowledge of the object. Take a particular case if you want to understand what I have just said, and then apply the theory to all objects in the same way. There is, for example, something called a circle, whose name is the very word I just now uttered. In the second place there is a description of it, made up of nouns and verbs. The description of the object whose name is round and circumference and circle would be : that which has everywhere the same distance between the extremities and the middle. In the third place there is the object which is drawn and erased and turned on the lathe and destroyed—processes which the real circle, in relation to which these other circles exist, can in no wise suffer, being different from them. In the fourth place there are knowledge and understanding and correct opinion about them—all of which must be posited as one thing more, inasmuch as it is found not in sounds nor in the shapes of bodies but in souls, whereby it manifestly differs in nature both from the real circle and from the aforesaid three. Of these understanding approaches nearest to the fifth in kinship and likeness, while the others are more distant. . . . Every circle drawn or turned on a lathe in practice abounds in the opposite to the fifth—for it everywhere touches the straight, while the real circle, we maintain, contains in itself neither more nor less of the opposite nature. The name, we maintain, is in no case stable ; there is nothing to prevent the things now called round from being called straight, and the straight round ; and those

οὐδὲν ἧττον βεβαίως ἕξειν τοῖς μεταθεμένοις καὶ ἐναντίως καλοῦσιν.

Aristot. *Met.* A 5, 987 b 14-18

Ἔτι δὲ παρὰ τὰ αἰσθητὰ καὶ τὰ εἴδη τὰ μαθηματικὰ τῶν πραγμάτων εἶναί φησι μεταξύ, διαφέροντα τῶν μὲν αἰσθητῶν τῷ ἀΐδια καὶ ἀκίνητα εἶναι, τῶν δ' εἰδῶν τῷ τὰ μὲν πόλλ' ἄττα ὅμοια εἶναι τὸ δ' εἶδος αὐτὸ ἓν ἕκαστον μόνον.

(c) The "Diorismos" in the "Meno"

Plat. *Meno* 86 E–87 B

Λέγω δὲ τὸ ἐξ ὑποθέσεως ὧδε, ὥσπερ οἱ γεωμέτραι πολλάκις σκοποῦνται, ἐπειδάν τις ἔρηται αὐτούς, οἷον περὶ χωρίου, εἰ οἷόν τε ἐς τόνδε τὸν κύκλον τόδε τὸ χωρίον τρίγωνον ἐνταθῆναι, εἴποι ἄν τις ὅτι "οὔπω οἶδα εἰ ἔστι τοῦτο τοιοῦτον, ἀλλ' ὥσπερ μέν τινα ὑπόθεσιν προὔργου οἶμαι ἔχειν πρὸς τὸ πρᾶγμα τοιάνδε. εἰ μέν ἐστι τοῦτο τὸ χωρίον τοιοῦτον, οἷον παρὰ τὴν δοθεῖσαν αὐτοῦ γραμμὴν παρατείναντα ἐλλείπειν τοιούτῳ χωρίῳ, οἷον ἂν αὐτὸ τὸ παρατεταμένον ᾖ, ἄλλο τι συμβαίνειν μοι δοκεῖ, καὶ ἄλλο αὖ, εἰ ἀδύνατόν ἐστι ταῦτα παθεῖν· ὑποθέμενος οὖν ἐθέλω εἰπεῖν σοι

who transpose them and use them in the opposite way will find them no less stable than they are now.

Aristotle, *Metaphysics* A 5, 987 b 14-18

Again, he [Plato] said that besides perceptible objects and forms there are the objects of mathematics, which occupy an intermediate position ; they differ from perceptible objects in being eternal and unchangeable, and from forms in that there are many alike, while the form itself is in each case unique.

(c) The " Diorismos " in the " Meno "

Plato, *Meno* 86 E–87 B

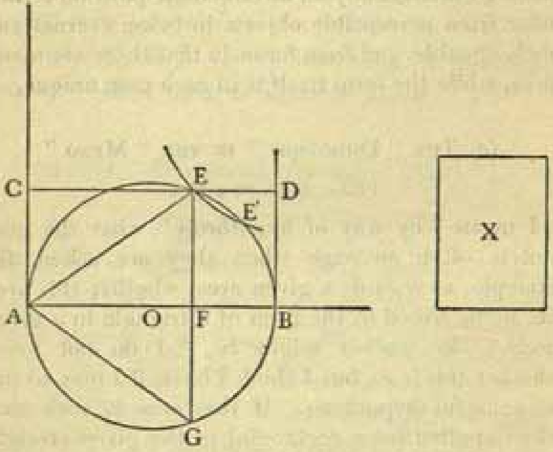
I mean " by way of hypothesis " what the geometers often envisage when they are asked, for example, as regards a given area, whether this area can be inscribed in the form of a triangle in a given circle. The answer might be, " I do not know whether this is so, but I think I have, if I may so put it, a useful hypothesis. If this area is such that when applied [as a rectangle] to the given straight line [a] in the circle it is deficient by a figure [rectangle] similar to that which is applied, then one result seems to me to follow, while another result follows if what I have described is not possible. Accordingly, by laying down a hypothesis I am willing to tell you

[a] " The given straight line " can only be the diameter. The " application " of areas so as to be " deficient " in a given way is explained above, pp. 186-187.

τὸ συμβαῖνον περὶ τῆς ἐντάσεως αὐτοῦ εἰς τὸν κύκλον, εἴτε ἀδύνατον εἴτε μή."

---

* If AB is the diameter of a circle of centre O, and E is a point on the circumference, and the rectangles ACEF,



FBDE are completed, and the chords EFG, AG are drawn, then the rectangle ACEF is "applied" to the straight line AB and "falls short" by the rectangle FBDE which is similar to the "applied" rectangle, for AF : FE = EF : FB. Moreover AEG is an isosceles triangle equal in area to the rectangle ACEF.

In order, therefore, to inscribe in the circle an isosceles

what is the conclusion about the inscribing of the area in the circle, whether it is impossible or not." [a]

triangle equal to a given area X we have to find a point E on the circumference of the circle such that if EF be dropped perpendicular to AB

the rectangle AF . FE = the given area X.

Clearly E lies on a rectangular hyperbola of which AB, AC are asymptotes. If b^2 is equal to the given area, the equation of the hyperbola referred to its asymptotes as axes is $xy = b^2$. For a real solution it is necessary that b^2 should not be greater than the equilateral triangle inscribed in the circle, *i.e.*, not greater than $3\sqrt{3} \cdot \frac{a^2}{4}$, where a is the radius of the circle. If b^2 is equal to this area, the hyperbola touches the circle and there is only one solution. If b^2 is greater than this area, the hyperbola does not touch, and there is no solution. If b^2 is less than this area, the hyperbola cuts the circle in two points E, E′, giving two solutions. It is to these facts that Plato refers.

The passage is an example of a διορισμός giving the conditions for the possibility of the solution of a problem. Proclus is therefore in error when he says that Leon, the pupil of Neoclides, who was younger than Plato, "invented διορισμοί" (*supra*, p. 150).

The above interpretation was first given by E. F. August in 1829. It was independently discovered by S. H. Butcher in *Journal of Philology*, xvii., pp. 219-225 and is accepted by Heath (*H.G.M.* i. 298-303), whose exposition I have closely followed. Many other explanations have been offered, the best known being that of Adolph Benecke (*Ueber die geometrische Hypothesis in Platons Menon*).

(*d*) The Nuptial Number

Plat. *Rep.* viii. 546 B-D.

Ἔστι δὲ θείῳ μὲν γεννητῷ περίοδος ἣν ἀριθμὸς περιλαμβάνει τέλειος, ἀνθρωπείῳ δὲ ἐν ᾧ πρώτῳ αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι, τρεῖς ἀποστάσεις, τέτταρας δὲ ὅρους λαβοῦσαι ὁμοιούντων τε καὶ ἀνομοιούντων καὶ αὐξόντων καὶ φθινόντων, πάντα προσήγορα καὶ ῥητὰ πρὸς ἄλληλα ἀπέφηναν· ὧν ἐπίτριτος πυθμὴν πεμπάδι συζυγεὶς δύο ἁρμονίας παρέχεται τρὶς αὐξηθείς, τὴν μὲν ἴσην ἰσάκις, ἑκατὸν τοσαυτάκις, τὴν δὲ ἰσομήκη μὲν τῇ, προμήκη δέ, ἑκατὸν μὲν ἀριθμῶν ἀπὸ διαμέτρων ῥητῶν πεμπάδος, δεομένων ἑνὸς ἑκάστων, ἀρρήτων δὲ δυοῖν, ἑκατὸν δὲ κύβων τριάδος.

---

[a] The passage is included here because of several interesting points for the history of Greek mathematics. Plato's language is so fancifully phrased that a completely satisfactory solution is difficult to get. The literature which has grown round this "nuptial number" is vast, but the most satisfying discussions are those by Adam, *The Republic of Plato* ii., pp. 204-208, 264-312, and A. G. Laird, *Plato's Geometrical Number and the Comment of Proclus*.

[b] δυναστευόμεναι is a ἅπαξ λεγόμενον, and its meaning is uncertain. A straight line is said δύνασθαι ("to be capable of") an area when the square on it is equal to the area. Hence δυναμένη should mean the side of a square, as it does in Eucl. x. Def. 4. δυναστευομένη is a kind of passive of δυναμένη, meaning presumably that of which the δυναμένη is capable, and so could mean the square itself. It is

(*d*) The Nuptial Number

Plato, *Republic* viii. 546 b-d [a]

The divine race has a cycle comprehended by a perfect number, but the number of the human race's cycle is the first in which root and square increases,[b] forming three intervals and four terms of elements that make like and unlike and wax and wane, show all things agreeable and rational towards one another. The base of these things, the four-three joined with five, when thrice increased furnishes two harmonies, the one a square, so many times a hundred, the other a rectangle, one of its sides being a hundred of the numbers from the rational diameters of five, each diminished by one (or a hundred of the numbers from the irrational diameters of five, each diminished by two), the other side being a hundred of the cubes of three.[c]

temerarious to try and get a precise meaning out of αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι, and perhaps we should not inquire too closely into what is more mystical than mathematical. Laird thinks it means "if a square is equal to a rectangle."

[c] The chief mathematical interest of the passage lies in the part most easy to decipher, that about the two "harmonies." The "irrational diameter of five" is the diagonal of a side of square 5, *i.e.* $\sqrt{50}$. The "rational diameter" of five is the nearest integer to the "irrational diameter," *i.e.* $\sqrt{50-1}$. The "number" from the "rational" or "irrational" diameter is the square. A "hundred of the numbers from the rational diameter of five, each diminished by one" is therefore $100 \times (49-1) = 4800$; and the same number is expressed as "a hundred of the numbers from the irrational diameter of five, each diminished by two," for this is $100 \times (50-2) = 4800$. This number gives one side of the oblong and the other is "a hundred of the cubes of three," or $100 \times 27 = 2700$. The rectangle of which these

(e) Generation of Numbers

Plat. *Epin.* 990 c–991 b

Διὸ μαθημάτων δέον ἂν εἴη· τὸ δὲ μέγιστόν τε καὶ πρῶτον καὶ ἀριθμῶν αὐτῶν, ἀλλ' οὐ σώματα ἐχόντων, ἀλλὰ ὅλης τῆς τοῦ περιττοῦ τε καὶ ἀρτίου γενέσεώς τε καὶ δυνάμεως, ὅσην παρέχεται πρὸς τὴν τῶν ὄντων φύσιν. ταῦτα δὲ μαθόντι τούτοις ἐφεξῆς ἐστιν ὃ καλοῦσι μὲν σφόδρα γελοῖον ὄνομα γεωμετρίαν, τῶν οὐκ ὄντων δὲ ὁμοίων ἀλλήλοις φύσει ἀριθμῶν ὁμοίωσις πρὸς τὴν τῶν ἐπιπέδων μοῖραν γεγονυῖά ἐστι διαφανής· ὃ δὴ θαῦμα οὐκ ἀνθρώπινον ἀλλὰ γεγονὸς θεῖον φανερὸν ἂν γίγνοιτο τῷ δυναμένῳ συννοεῖν. μετὰ δὲ ταύτην τοὺς τρὶς

---

are sides is therefore $4800 \times 2700 = 12{,}960{,}000$, and this is 3600^2, which is the other "harmony."

These "rational" and "irrational" diameters are a clear reference to the "side-" and "diameter- numbers" of the Pythagoreans, for which see *supra*, pp. 132-139.

There is fairly widespread agreement that the geometrical number is $12{,}960{,}000 = 3600^2 = 4800 \times 2700$, but on the method by which this number is reached the widest divergence exists. Hultsch and Adam suppose that two numbers are obtained, one in the first sentence down to ἀπέφηναν, the other (12,960,000) in the remainder of the passage. Both agree that the first number is 216, but Hultsch obtains it as $2^3 \times 3^3$ and Adam as $3^3 + 4^3 + 5^3$. Hultsch then takes "the four-three joined with a five" to mean $4 + 3 + 5 = 12$, which is then multiplied by three (τρὶς αὐξηθείς), giving 36, and as this has to be taken "so many times a hundred" we get 3600 as the side of the square which is one of the "harmonies," and therefore the final number is 3600^2. Adam takes "the four-three joined with a five" to be $3 \times 4 \times 5 = 60$, and τρὶς αὐξηθείς to mean multiplied by itself three times (*i.e.* raised to the fourth power, which gives us immediately $60^4 = 3600^2$). Laird, on the other hand, believes there is only one number

(e) Generation of Numbers

Plato, *Epinomis* 990 c–991 b

There will therefore be need of studies[a]: the first and most important is of numbers in themselves, not of corporeal numbers, but of the whole genesis of the odd and even, and the greatness of their influence on the nature of things. When the student has learnt these matters there comes next in order after them what they call by the very ridiculous name of geometry, though it proves to be an evident likening, with reference to planes, of numbers not like one another by nature[b]; and that this is a marvel not of human but of divine origin will be clear to him who is able to understand. And after this the numbers

indicated (which he agrees in thinking to be $3600^2 = 4800 \times 2700$). He maintains, with the help of Proclus, that the first sentence gives a general method of forming "harmonies" which is then applied to the *triangle* of sides 3, 4 and 5 to give the geometrical number. The application gives the series 27, 36, 48, 64 (with four terms and three intervals), and the first three numbers multiplied by 100 give the elements of the geometrical number, $3600^2 = 2700 \times 4800$. Each solution has merits, but each raises problems which it is impossible to discuss here. However, we may be fairly confident that the final number obtained is 12,960,000.

[a] In Plato the word μάθημα is used generally of any study, but the particular subjects here mentioned are all mathematical, and the word was already getting the special significance which it attained in Aristotle's time.

[b] The most likely explanation of "numbers not like one another by nature" is "numbers incommensurable with each other"; drawn as two lines in a plane, *e.g.* as the side and diagonal of a square, they are made like to one another by the geometer's art, in that there is no outward difference between them as there is between an integer and an irrational number.

ηὐξημένους καὶ τῇ στερεᾷ φύσει ὁμοίους, τοὺς δὲ ἀνομοίους αὖ γεγονότας ἑτέρᾳ τέχνῃ ὁμοιοῖ, ταύτῃ ἣν δὴ στερεομετρίαν ἐκάλεσαν οἱ προστυχεῖς αὐτῇ γεγονότες· ὃ δὲ θεῖόν τ' ἐστὶ καὶ θαυμαστὸν τοῖς ἐγκαθορῶσί τε καὶ διανοουμένοις, ὡς περὶ τὸ διπλάσιον ἀεὶ στρεφομένης τῆς δυνάμεως καὶ τῆς ἐξ ἐναντίας ταύτῃ καθ' ἑκάστην ἀναλογίαν εἶδος καὶ γένος ἀποτυποῦται πᾶσα ἡ φύσις. ἡ μὲν δὴ πρώτη τοῦ διπλασίου κατ' ἀριθμὸν ἓν πρὸς δύο κατὰ λόγον φερομένη, διπλάσιον δὲ ἡ κατὰ δύναμιν οὖσα· ἡ δ' εἰς τὸ στερεόν τε καὶ ἁπτὸν πάλιν αὖ διπλάσιον, ἀφ' ἑνὸς εἰς ὀκτὼ διαπορευθεῖσα· ἡ δὲ διπλασίου μὲν εἰς μέσον, ἴσως δὲ τοῦ ἐλάττονος πλέον ἔλαττόν τε τοῦ μείζονος, τὸ δ' ἕτερον τῷ αὐτῷ μέρει τῶν ἄκρων αὐτῶν ὑπερέχον τε καὶ ὑπερεχόμενον· ἐν μέσῳ δὲ τοῦ ἓξ πρὸς τὰ δώδεκα συνέβη τό τε ἡμιόλιον καὶ ἐπίτριτον· τούτων αὐτῶν

---

[a] These are probably cubes of integers.

[b] These will be numbers with irrational cube roots.

[c] What has been said about lines in the plane applies also to lines in three dimensions. Numbers incommensurable with each other, such as 1 and $\sqrt[3]{2}$, are made like when one is represented as the side of a unit cube and the other as the side of a cube twice as great. We know that this problem of doubling the cube was brought to Plato's notice (*supra*, pp. 258-259). The past tense suggests that Plato had in mind certain definite προστυχεῖς who coined the word στερεομετρία: the Pythagoreans, Theaetetus, Democritus and Eudoxus had all advanced the science.

[d] What follows cannot be translated literally, and it is more than likely that the text is corrupt, or that it has reached us unrevised from Plato's first draft. But the general sense is clear. Successive multiplication of 1 by 2

thrice increased and like to the solid nature,[a] and those again which have been made unlike,[b] he likens by another art, namely, that which its adepts called stereometry[c]; and a divine and marvellous thing it is to those who contemplate it and reflect how the whole of nature is impressed with species and kind according to each proportion as power and its converse continually turn about the double.[d] First the double operates on the number 1 by simple multiplication so as to give 2, and a second double yields the square; by further doubling we reach the solid and tangible, the process having gone from 1 to 8. Then comes the application of the double to give the mean which is as much greater than the less as it is less than the greater, and the other mean is that which exceeds and is exceeded by the same part of the extremes; between 6 and 12 come both the *sesquialter* [9] and the *sesquitertius* [8]; turning between these two, to

gives the series 1, 2, 4, 8, which represent a point, a line, a square and a cube. This is a series in geometric progression, 2 being a geometrical mean between 1 and 4, and 4 a geometrical mean between 2 and 8. Two other means were known to the Pythagoreans (*supra*, pp. 110-115)—and the whole passage is thoroughly Pythagorean—the arithmetic and the harmonic. The arithmetic mean is equidistant between the two terms; the harmonic exceeds one term, and is exceeded by the other, by the same fraction of each term. Thus the arithmetic mean between 1 and 2 is $\frac{3}{2}$ and the harmonic mean is $\frac{4}{3}$; clearing of fractions, the arithmetic mean between 6 and 12 is 9 and the harmonic mean 8.

[d] "Power and its converse"—ἡ δύναμις καὶ ἡ ἐξ ἐναντίας ταύτῃ—I take to mean "number and its reciprocal"; we have to multiply by 2 to get the series 1, 2, 4, 8 and then take $\frac{1}{2}$ of $6 + 12$ to get the arithmetic mean.

ἐν τῷ μέσῳ ἐπ' ἀμφότερα στρεφομένη τοῖς ἀνθρώποις σύμφωνον χρείαν καὶ σύμμετρον ἀπενείματο παιδιᾶς ῥυθμοῦ τε καὶ ἁρμονίας χάριν, εὐδαίμονι χορείᾳ Μουσῶν δεδομένη.

[a] The reference to the choir of the Muses makes it clear, in my opinion, that the number 9 is referred to, though the construction of the sentence does not necessarily involve it. So W. R. M. Lamb in the Loeb version of the *Epinomis*, p. 482.

[b] The whole passage should be compared with *Timaeus*, 35 B—36 B (see R. G. Bury's notes in the Loeb version, pp. 66-71, or A. E. Taylor, *A Commentary on Plato's Timaeus*, pp. 136-137). There Plato writes down the series 1, 2, 4, 8 and 1, 3, 9, 27, and then fills up the intervals between these

one side or the other, this power [9] [a] furnished men with concord and symmetry for the purpose of rhythm and harmony in their pastimes, and has been given to the blessed dance of the Muses.[b]

numbers with arithmetic and harmonic means so as to get a series of 34 terms, 1, $\frac{9}{8}$, $\frac{81}{64}$, $\frac{4}{3}$, $\frac{3}{2}$, $\frac{27}{16}$, $\frac{243}{128}$, 2 . . . 27, which is intended to represent the notes of a musical scale having a compass of four octaves and a major "sixth."

Much prominence is given to this passage from the *Epinomis* by A. E. Taylor, *Mind*, xxxv., pp. 419-440, 1926, *ibid.* xxxvi., pp. 12-33, 1927, and D'Arcy Wentworth Thompson, *ibid.* xxxviii., pp. 43-55, 1929.

For a further discussion of this side of Plato's philosophy see Julius Stenzel, *Zahl und Gestalt bei Platon und Aristoteles* (Leipzig, 1924).

XIII. EUDOXUS OF CNIDOS

XIII. EUDOXUS OF CNIDOS

(a) Theory of Proportion

Schol. i. in Eucl. *Elem.* v., Eucl. ed. Heiberg v. 280. 1-9

Σκοπὸς τῷ πέμπτῳ βιβλίῳ περὶ ἀναλογιῶν διαλαβεῖν. . . . τὸ δὲ βιβλίον Εὐδόξου τινὲς εὕρεσιν εἶναι λέγουσι τοῦ Πλάτωνος διδασκάλου.

(b) Volume of Cone and Pyramid

Archim. *De Sphaera et Cyl.* i., Pref., Archim. ed. Heiberg i. 4. 2-13

Διόπερ οὐκ ἂν ὀκνήσαιμι ἀντιπαραβαλεῖν αὐτὰ πρός τε τὰ τοῖς ἄλλοις γεωμέτραις τεθεωρημένα καὶ πρὸς τὰ δόξαντα πολὺ ὑπερέχειν τῶν ὑπὸ Εὐδόξου περὶ τὰ στερεὰ θεωρηθέντων, ὅτι πᾶσα πυραμὶς τρίτον ἐστὶ μέρος πρίσματος τοῦ βάσιν ἔχοντος τὴν αὐτὴν τῇ πυραμίδι καὶ ὕψος ἴσον, καὶ ὅτι πᾶς κῶνος τρίτον μέρος ἐστὶν τοῦ κυλίνδρου τοῦ βάσιν ἔχοντος τὴν αὐτὴν τῷ κώνῳ καὶ ὕψος ἴσον· καὶ γὰρ τούτων προυπαρχόντων φυσικῶς περὶ ταῦτα τὰ σχήματα, πολλῶν πρὸ Εὐδόξου

XIII. EUDOXUS OF CNIDOS[a]

(a) Theory of Proportion

Euclid, *Elements* v., Scholium i., Eucl. ed. Heiberg v. 280. 1-9

The aim of the fifth [book of the *Elements*] is the treatment of proportionals. . . . Some say that the book is the discovery of Eudoxus, the pupil of Plato.

(b) Volume of Cone and Pyramid

Archimedes, *On the Sphere and Cylinder*, Preface to Book i., Archim. ed. Heiberg i. 4. 2-13

For this reason I cannot feel any hesitation in setting these [theorems] side by side both with the investigations of other geometers and with those of the theorems of Eudoxus on solids which seem to stand out pre-eminently, namely, that any pyramid is a third part of the prism having the same base as the pyramid and equal height, and that any cone is a third part of the cylinder having the same base as the cone and equal height; for though these properties were naturally inherent in these figures all along, yet

[a] Eudoxus lived from about 408 to 355 B.C. For Proclus's notice of him, see *supra*, pp. 150-153.

γεγενημένων ἀξίων λόγου γεωμετρῶν συνέβαινεν ὑπὸ πάντων ἀγνοεῖσθαι μηδ' ὑφ' ἑνὸς κατανοηθῆναι

(c) Theory of Concentric Spheres

Aristot. *Met.* Λ 8, 1073 b 17-32

Εὔδοξος μὲν οὖν ἡλίου καὶ σελήνης ἑκατέρου τὴν φορὰν ἐν τρισὶν ἐτίθετ' εἶναι σφαίραις, ὧν τὴν μὲν πρώτην τὴν τῶν ἀπλανῶν ἄστρων εἶναι, τὴν δὲ δευτέραν κατὰ τὸν διὰ μέσων τῶν ζῳδίων, τὴν δὲ τρίτην κατὰ τὸν λελοξωμένον ἐν τῷ

---

[a] In his preface to the *Method* (see *supra*, p. 230) Archimedes says that Democritus enunciated these theorems, but without proof. It may safely be inferred from Archimedes' preface to the *Quadrature of the Parabola* (Archim. ed. Heiberg ii. 264. 9-22) that Eudoxus used for the proof a lemma equivalent to Euclid x. 1 (*infra*, pp. 452-455), and that the credit belongs to him for having made the *exhaustion* of an area by means of inscribed polygons a regular method in Greek geometry: to some extent he had been preceded by Antiphon and Hippocrates.

[b] We are told by Simplicius, on the authority of Eudemus, that Plato set astronomers the problem of finding what are the uniform and ordered movements which will "save the phenomena" of the planetary motions, and that Eudoxus was the first of the Greeks to concern himself with hypotheses of this sort. Eudoxus believed that the motion of the sun, moon and planets could be accounted for by a combination of circular movements, a view which remained unchallenged till Kepler. To account for the motion of the sun and moon he needed to use only three concentric spheres, but the motion of the planets required in each case four concentric spheres, the common centre being the centre of the earth. The spheres were of different sizes, one enclosing the other. Each planet was attached to a point on the equator of the innermost sphere, so that by the motion of this sphere alone the planet would describe a circle. But the poles of this

they were in fact unknown to the many competent geometers who lived before Eudoxus and had not been noticed by anyone.[a]

(c) Theory of Concentric Spheres

Aristotle, *Metaphysics* Λ 8, 1073 b 17-32

Eudoxus assumed that the motion both of the sun and of the moon takes place on three spheres,[b] of which the first is that of the fixed stars, the second moves about the circle which passes through the middle of the signs of the zodiac, and the third moves about

sphere were not fixed, themselves moving on a larger sphere rotating about two different poles. The poles of this second sphere similarly lay on a third larger sphere moving about a different set of poles, and the poles of the third sphere on yet a fourth, moving about another set of poles. Each sphere rotated uniformly, but its speed was peculiar to itself. For the sun and moon only three spheres were needed, the two largest being the same as for the planets The outermost circle (which comes first in the description by Aristotle and Simplicius), moving from east to west in twenty-four hours, reproduces the daily motion of the fixed stars. The second moves from west to east about an axis perpendicular to the plane of the zodiac circle (ecliptic), its equator accordingly revolving in the plane of the zodiac.

The subject belongs as much to Greek astronomy as to Greek mathematics, and for fuller information the reader is referred to the classic paper of Schiaparelli, *Le sfere omocentriche di Eudosso, di Callippo e di Aristotele* (Milan, 1875), to the works of Sir Thomas Heath (*Aristarchus of Samos*, pp. 193-224, *Greek Astronomy*, pp. 65-70, *H.G.M.* i. 329-335), and to W. D. Ross, *Aristotle's Metaphysics*, vol. ii., pp. 384-394. But Eudoxus's system of concentric rotating spheres is a geometrical *tour de force* of the highest order, and must find some notice here. In all the history of science there are few hypotheses that bear so unmistakably the stamp of genius.

πλάτει τῶν ζῳδίων (ἐν μείζονι δὲ πλάτει λελοξῶσθαι καθ' ὃν ἡ σελήνη φέρεται ἢ καθ' ὃν ὁ ἥλιος). τῶν δὲ πλανωμένων ἄστρων ἐν τέτταρσιν ἑκάστου σφαίραις, καὶ τούτων δὲ τὴν μὲν πρώτην καὶ δευτέραν τὴν αὐτὴν εἶναι ἐκείναις (τήν τε γὰρ τῶν ἀπλανῶν τὴν ἁπάσας φέρουσαν εἶναι, καὶ τὴν ὑπὸ ταύτῃ τεταγμένην καὶ κατὰ τὸν διὰ μέσων τῶν ζῳδίων τὴν φορὰν ἔχουσαν κοινὴν ἁπασῶν εἶναι), τῆς δὲ τρίτης ἁπάντων τοὺς πόλους ἐν τῷ διὰ μέσων τῶν ζῳδίων εἶναι, τῆς δὲ τετάρτης τὴν φορὰν κατὰ τὸν λελοξωμένον πρὸς τὸν μέσον ταύτης· εἶναι δὲ τῆς τρίτης σφαίρας τοὺς πόλους τῶν μὲν ἄλλων ἰδίους, τοὺς δὲ τῆς Ἀφροδίτης καὶ τοῦ Ἑρμοῦ τοὺς αὐτούς.

Simpl. *in De caelo* ii. 12 (Aristot. 293 a 4), ed. Heiberg 496. 23–497. 5

Ἡ δὲ τρίτη σφαῖρα τοὺς πόλους ἔχουσα ἐπὶ τοῦ ἐν τῇ δευτέρᾳ διὰ μέσων τῶν ζῳδίων ἀπὸ μεσημβρίας τε πρὸς ἄρκτον στρεφομένη καὶ ἀπ' ἄρκτου πρὸς μεσημβρίαν συνεπιστρέψει τὴν τετάρτην καὶ ἐν αὐτῇ τὸν ἀστέρα ἔχουσαν καὶ δὴ τῆς κατὰ πλάτος κινήσεως ἕξει τὴν αἰτίαν· οὐ μὴν αὐτὴ μόνη· ὅσον γὰρ ἐπὶ ταύτῃ καὶ πρὸς τοὺς πόλους τοῦ διὰ μέσων τῶν ζῳδίων ἧκεν ἂν ὁ ἀστὴρ καὶ πλησίον τῶν τοῦ κόσμου πόλων ἐγίνετο· νυνὶ δὲ ἡ τετάρτη σφαῖρα περὶ τοὺς τοῦ ⟨τοῦ⟩[1] ἀστέρος λοξοῦ κύκλου στρεφομένη πόλους ἐπὶ τἀναντία τῇ τρίτῃ ἀπ' ἀνατολῶν ἐπὶ δυσμὰς καὶ ἐν ἴσῳ χρόνῳ

[1] τοῦ τοῦ Heiberg.

[a] *i.e.* the equator of the third sphere.
[b] *i.e.* Venus and Mercury.

a circle latitudinally inclined to the zodiac circle (the circle in which the moon moves having a greater latitudinal inclination than that of the sun). The motion of the planets he assumed to take place in each case on four spheres; of these the first and second are the same as for the sun and moon (the first being the sphere of the fixed stars which carries all the spheres with it, and the second, next in order to it, being the sphere about the circle through the middle of the signs of the zodiac which is common to all the planets); the third is, in all cases, a sphere with its poles on the circle through the middle of the signs of the zodiac; and the fourth moves about a circle inclined to the middle circle[a] of the third sphere; the poles of the third sphere are different for all the planets except Aphrodite and Hermes,[b] but for these the poles are the same.

Simplicius, *Commentary on Aristotle's De caelo* ii. 12 (293 a 4), ed. Heiberg 496. 23–497. 5

The third sphere, which has its poles on the great circle of the second sphere passing through the middle of the signs of the zodiac, and which turns from south to north and from north to south, will carry round with it the fourth sphere, which has the planet attached to it, and will moreover be the cause of the planet's latitudinal movement. But not the third sphere only; for, in so far as it was on this sphere only, the planet would have reached the poles of the zodiac circle, and would have drawn near to the poles of the universe; but as matters are, the fourth sphere, which turns about the poles of the inclined circle carrying the planet and rotates in a sense opposite to the third, that is, from east to west, but in the same

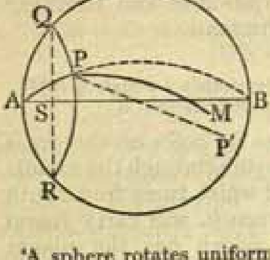
τὴν στροφὴν αὐτῶν ποιουμένη τό τε ἐπὶ πλέον ὑπερβάλλειν τὸν διὰ μέσων τῶν ζῳδίων παραιτήσεται καὶ τὴν λεγομένην ὑπὸ Εὐδόξου ἱπποπέδην περὶ τὸν αὐτὸν τουτονὶ κύκλον τῷ ἀστέρι γράφειν παρέξεται, ὥστε, ὁπόσον τὸ τῆς γραμμῆς ταύτης πλάτος, τοσοῦτον καὶ ὁ ἀστὴρ εἰς πλάτος δόξει παραχωρεῖν, ὅπερ ἐγκαλοῦσι τῷ Εὐδόξῳ.

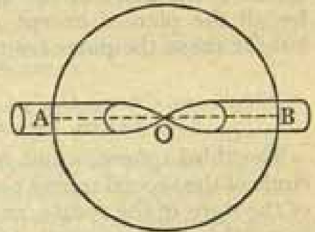
---

[a] *i.e.* by the planet.

[b] *i.e.* "horse-fetter."

[c] Schiaparelli works out in detail the motion of a planet subject only to the rotations of the third and fourth spheres. The problem in its simplest expression, he says, is this:





"A sphere rotates uniformly about the fixed diameter AB. P, P′ are opposite poles on this sphere, and a second sphere concentric with the first rotates uniformly about PP′ in the same time as the former sphere takes to turn about AB, but in the opposite direction. M is a point on the second sphere equidistant from the poles P, P′ (that is to say, M is a point on the equator of the second sphere). It is required to find the path of M." Schiaparelli found a solution by means of seven geometrical propositions which Eudoxus could have known, and he proved that the path described by M was like a figure-of-eight on the surface of the sphere (see second figure). This curve, which Schiaparelli called a

period, will prevent any excessive deviation [a] from the circle through the middle of the signs of the zodiac, and will constrain the planet to describe about the same zodiac circle the curve called by Eudoxus the *hippopede*,[b] so that the breadth of this curve measures the apparent latitudinal motion of the planet, a view for which Eudoxus has been attacked.[c]

"spherical lemniscate," agrees with Eudoxus's description of it as a *hippopede* (horse-fetter). It is the intersection of the sphere with a certain cylinder touching it internally at the double point O, namely, a cylinder with diameter equal to AS, the *sagitta* of the diameter of the small circle of the sphere on which P revolves.

For the proof of these statements the reader must be referred to Schiaparelli's paper. An analytical expression is given by Norbert Herz in *Geschichte der Bahnbestimmung von Planeten und Kometen*, Part i., pp. 20, 21, and reproduced by Heath, *Aristarchus of Samos*, pp. 204-205, with further details.

Summing up, Heath says (*Aristarchus of Samos*, p. 211): "For the sun and moon the hypothesis of Eudoxus sufficed to explain adequately enough the principal phenomena, except the irregularities due to the eccentricities, which were either unknown to Eudoxus or neglected by him. For Jupiter and Saturn, and to some extent for Mercury also, the system was capable of giving on the whole a satisfactory explanation of their motion in longitude, their stationary points and their retrograde motions; for Venus it was unsatisfactory, and it failed altogether in the case of Mars. The limits of motion in latitude represented by the various *hippopedes* were in tolerable agreement with observed facts, although the periods of their deviations and their places in the cycle were quite wrong. But, notwithstanding the imperfections of the system of homocentric spheres, we cannot but recognize in it a speculative achievement which was worthy of the great reputation of Eudoxus and all the more deserving of admiration because it was the first attempt at a scientific explanation of the apparent irregularities of the motions of the planets."

XIV. ARISTOTLE

XIV. ARISTOTLE

(a) First Principles

Aristot. *Anal. Post.* i. 10, 76 a 30–77 a 2

Λέγω δ' ἀρχὰς ἐν ἑκάστῳ γένει ταύτας, ἃς ὅτι ἔστι μὴ ἐνδέχεται δεῖξαι. τί μὲν οὖν σημαίνει καὶ τὰ πρῶτα καὶ τὰ ἐκ τούτων, λαμβάνεται· ὅτι δ' ἔστι, τὰς μὲν ἀρχὰς ἀνάγκη λαμβάνειν, τὰ δ' ἄλλα δεικνύναι, οἷον τί μονὰς ἢ τί τὸ εὐθὺ καὶ τρίγωνον· εἶναι δὲ τὴν μονάδα λαβεῖν καὶ μέγεθος, τὰ δ' ἕτερα δεικνύναι.

Ἔστι δ' ὧν χρῶνται ἐν ταῖς ἀποδεικτικαῖς ἐπιστήμαις τὰ μὲν ἴδια ἑκάστης ἐπιστήμης τὰ δὲ κοινά, κοινὰ δὲ κατ' ἀναλογίαν, ἐπεὶ χρήσιμόν γε ὅσον ἐν τῷ ὑπὸ τὴν ἐπιστήμην γένει.

Ἴδια μὲν οἷον γραμμὴν εἶναι τοιανδί, καὶ τὸ εὐθύ, κοινὰ δὲ οἷον τὸ ἴσα ἀπὸ ἴσων ἂν ἀφέλῃ, ὅτι ἴσα τὰ λοιπά. ἱκανὸν δ' ἕκαστον τούτων ὅσον ἐν τῷ γένει· ταὐτὸ γὰρ ποιήσει, κἂν μὴ κατὰ πάντων

---

[a] Aristotle interspersed his writings with illustrations from mathematics, and as he lived just before Euclid he throws valuable light on the transformation which Euclid effected. A large number of the mathematical passages in Aristotle's works are translated, with valuable notes, in Sir Thomas Heath's posthumous book *Mathematics in Aristotle*.

XIV. ARISTOTLE [a]

(a) First Principles

Aristotle, *Posterior Analytics* i. 10, 76 a 30–77 a 2

I mean by the first principles in every genus those elements whose existence cannot be proved. The meaning both of these primary elements and of those deduced from them is assumed; in the case of first principles, their existence is also assumed, but in the case of the others deduced from them it has to be proved. Examples are given by the unit, the straight and triangular; for we must assume the existence of the unit and magnitude, but in the case of the others it has to be proved.

Of the first principles used in the demonstrative sciences some are peculiar to each science, and some are common, but common only by analogy, inasmuch as they are useful only in so far as they fall within the genus coming under the science in question.

Examples of peculiar first principles are given by the definitions of the line and the straight; common first principles are such as that, when equals are taken from equals, the remainders are equal. Only so much of these common first principles is needed as falls within the genus in question; for such a first principle will have the same force even though not

λάβῃ ἀλλ' ἐπὶ μεγεθῶν μόνον, τῷ δ' ἀριθμητικῷ ἐπ' ἀριθμῶν.

Ἔστι δ' ἴδια μὲν καὶ ἃ λαμβάνεται εἶναι, περὶ ἃ ἡ ἐπιστήμη θεωρεῖ τὰ ὑπάρχοντα καθ' αὑτά, οἷον μονάδας ἡ ἀριθμητική, ἡ δὲ γεωμετρία σημεῖα καὶ γραμμάς. ταῦτα γὰρ λαμβάνουσι τὸ εἶναι καὶ τοδὶ εἶναι. τὰ δὲ τούτων πάθη καθ' αὑτά, τί μὲν σημαίνει ἕκαστον, λαμβάνουσιν, οἷον ἡ μὲν ἀριθμητικὴ τί περιττὸν ἢ ἄρτιον ἢ τετράγωνον ἢ κύβος, ἡ δὲ γεωμετρία τί τὸ ἄλογον ἢ τὸ κεκλάσθαι ἢ νεύειν, ὅτι δ' ἔστι, δεικνύουσι διά τε τῶν κοινῶν καὶ ἐκ τῶν ἀποδεδειγμένων. καὶ ἡ ἀστρολογία ὡσαύτως. πᾶσα γὰρ ἀποδεικτικὴ ἐπιστήμη περὶ τρία ἐστίν, ὅσα τε εἶναι τίθεται (ταῦτα δ' ἐστὶ τὸ γένος, οὗ τῶν καθ' αὑτὰ παθημάτων ἐστὶ θεωρητική), καὶ τὰ κοινὰ λεγόμενα ἀξιώματα, ἐξ ὧν πρώτων ἀποδείκνυσι, καὶ τρίτον τὰ πάθη, ὧν τί σημαίνει ἕκαστον λαμβάνει. ἐνίας μέντοι ἐπιστήμας οὐδὲν κωλύει ἔνια τούτων παρορᾶν, οἷον τὸ γένος μὴ ὑποτίθεσθαι εἶναι, ἂν ᾖ φανερὸν ὅτι ἔστιν (οὐ γὰρ ὁμοίως δῆλον ὅτι ἀριθμός ἐστι καὶ ὅτι ψυχρὸν καὶ θερμόν), καὶ τὰ πάθη μὴ λαμβάνειν τί σημαίνει, ἂν ᾖ δῆλα· ὥσπερ οὐδὲ τὰ κοινὰ οὐ λαμβάνει τί σημαίνει τὸ ἴσα ἀπὸ ἴσων ἀφελεῖν, ὅτι γνώριμον. ἀλλ' οὐδὲν ἧττον τῇ γε φύσει τρία ταῦτά ἐστι, περὶ ὅ τε δείκνυσι καὶ ἃ δείκνυσι καὶ ἐξ ὧν.

Οὐκ ἔστι δ' ὑπόθεσις οὐδ' αἴτημα, ὃ ἀνάγκη

---

[a] Euclid does not define κεκλάσθαι "to be inflected," or νεύειν, "to verge." For an example of "inflection," see *supra*, pp. 358-359, and of "verging," *supra* pp. 242-245.

applied generally but only to magnitudes, or by the arithmetician only to numbers.

Also peculiar to a science are the first principles whose existence it assumes and whose essential attributes it investigates, for example, in arithmetic units, in geometry points and lines. Both their existence and their meaning are assumed. But of their essential attributes, only the meaning is assumed. For example, arithmetic assumes the meaning of odd and even, square and cube, geometry that of irrational or inflection or verging,[a] but their existence is proved from the common first principles and propositions already demonstrated. Astronomy proceeds in the same way. For indeed every demonstrative science has three elements: (1) that which it posits (the genus whose essential attributes it examines); (2) the so-called common axioms, which are the primary premisses in its demonstrations; (3) the essential attributes, whose meaning it assumes. There is nothing to prevent some sciences passing over some of these elements; for example, the genus may not be posited if it is obvious (the existence of number, for instance, and the existence of hot and cold are not similarly evident); or the meaning of the essential attributes might be omitted if that were clear. In the case of the common axioms, the meaning of taking equals from equals is not expressly assumed, being well known. Nevertheless in the nature of the case there are these three elements, that about which the demonstration takes place, that which is demonstrated and those premisses by which the demonstration is made.

That which necessarily exists from its very nature and which we must necessarily believe is neither

εἶναι δι' αὑτὸ καὶ δοκεῖν ἀνάγκη. οὐ γὰρ πρὸς τὸν ἔξω λόγον ἡ ἀπόδειξις, ἀλλὰ πρὸς τὸν ἐν τῇ ψυχῇ, ἐπεὶ οὐδὲ συλλογισμός. ἀεὶ γὰρ ἔστιν ἐνστῆναι πρὸς τὸν ἔξω λόγον, ἀλλὰ πρὸς τὸν ἔσω λόγον οὐκ ἀεί. ὅσα μὲν οὖν δεικτὰ ὄντα λαμβάνει αὐτὸς μὴ δείξας, ταῦτ', ἐὰν μὲν δοκοῦντα λαμβάνῃ τῷ μανθάνοντι, ὑποτίθεται, καὶ ἔστιν οὐχ ἁπλῶς ὑπόθεσις ἀλλὰ πρὸς ἐκεῖνον μόνον· ἂν δὲ ἢ μηδεμιᾶς ἐνούσης δόξης ἢ καὶ ἐναντίας ἐνούσης λαμβάνῃ τὸ αὐτὸ αἰτεῖται. καὶ τούτῳ διαφέρει ὑπόθεσις καὶ αἴτημα· ἔστι γὰρ αἴτημα τὸ ὑπεναντίον τοῦ μανθάνοντος τῇ δόξῃ, [ἢ] ὃ ἄν τις ἀποδεικτὸν ὂν λαμβάνῃ καὶ χρῆται μὴ δείξας.

Οἱ μὲν οὖν ὅροι οὐκ εἰσὶν ὑποθέσεις (οὐδὲ γὰρ εἶναι ἢ μὴ λέγονται), ἀλλ' ἐν ταῖς προτάσεσιν αἱ ὑποθέσεις. τοὺς δ' ὅρους μόνον ξυνίεσθαι δεῖ· τοῦτο δ' οὐχ ὑπόθεσις, εἰ μὴ καὶ τὸ ἀκούειν ὑπόθεσίν τις εἶναι φήσει. ἀλλ' ὅσων ὄντων τῷ ἐκεῖνα εἶναι γίνεται τὸ συμπέρασμα. οὐδ' ὁ γεωμέτρης ψευδῆ ὑποτίθεται, ὥσπερ τινὲς ἔφασαν, λέγοντες ὡς οὐ δεῖ τῷ ψεύδει χρῆσθαι, τὸν δὲ γεωμέτρην ψεύδεσθαι λέγοντα ποδιαίαν τὴν οὐ ποδιαίαν ἢ εὐθεῖαν τὴν γεγραμμένην οὐκ εὐθεῖαν οὖσαν. ὁ δὲ γεωμέτρης οὐδὲν συμπεραίνεται τῷ τήνδε εἶναι γραμμήν, ἣν αὐτὸς ἔφθεγκται, ἀλλὰ τὰ διὰ τούτων δηλούμενα. ἔτι τὸ αἴτημα καὶ ὑπόθεσις πᾶσα ἢ ὡς ὅλον ἢ ὡς ἐν μέρει, οἱ δ' ὅροι οὐδέτερον τούτων.

hypothesis nor postulate. For demonstration is a matter not of external discourse but of meditation within the soul, since syllogism is such a matter. And objection can always be raised to external discourse but not to inward meditation. That which is capable of proof but assumed by the teacher without proof is, if the pupil believes and accepts it, *hypothesis*, though it is not hypothesis absolutely but only in relation to the pupil; if the pupil has no opinion on it or holds a contrary opinion, the same assumption is a *postulate*. In this lies the distinction between hypothesis and postulate; for a postulate is contrary to the pupil's opinion, demonstrable, but assumed and used without demonstration.

The *definitions* are not hypotheses (for they do not assert either existence or non-existence), but it is in the premisses of a science that hypotheses lie. Definitions need only to be understood; and this is not hypothesis, unless it be contended that the pupil's hearing is also a hypothesis. But hypotheses lay down facts on whose existence depends the existence of the fact inferred. Nor are the geometer's hypotheses false, as some have maintained, urging that falsehood must not be used, and that the geometer is speaking falsely in saying that the line which he draws is a foot long or straight when it is neither a foot long nor straight. The geometer draws no conclusion from the existence of the particular line of which he speaks, but from what his diagrams represent. Furthermore, all hypotheses and postulates are either universal or particular, but a definition is neither.

(b) The Infinite

Aristot. *Phys.* Γ 6, 206 a 9-18

Ὅτι δ' εἰ μὴ ἔστιν ἄπειρον ἁπλῶς, πολλὰ ἀδύνατα συμβαίνει, δῆλον. τοῦ τε γὰρ χρόνου ἔσται τις ἀρχὴ καὶ τελευτή, καὶ τὰ μεγέθη οὐ διαιρετὰ εἰς μεγέθη, καὶ ἀριθμὸς οὐκ ἔσται ἄπειρος. ὅταν δὲ διωρισμένων οὕτως μηδετέρως φαίνηται ἐνδέχεσθαι, διαιρητοῦ δεῖ, καὶ δῆλον ὅτι πὼς μὲν ἔστιν πὼς δ' οὔ. λέγεται δὴ τὸ εἶναι τὸ μὲν δυνάμει τὸ δὲ ἐντελεχείᾳ, καὶ τὸ ἄπειρον ἔστι μὲν προσθέσει ἔστι δὲ καὶ διαιρέσει. τὸ δὲ μέγεθος ὅτι μὲν κατ' ἐνέργειαν οὐκ ἔστιν ἄπειρον, εἴρηται, διαιρέσει δ' ἐστίν· οὐ γὰρ χαλεπὸν ἀνελεῖν τὰς ἀτόμους γραμμάς· λείπεται οὖν δυνάμει εἶναι τὸ ἄπειρον.

Ibid. Γ 6, 206 b 3-12

Τὸ δὲ κατὰ πρόσθεσιν τὸ αὐτό ἐστί πως καὶ τὸ κατὰ διαίρεσιν· ἐν γὰρ τῷ πεπερασμένῳ κατὰ πρόσθεσιν γίγνεται ἀντεστραμμένως· ᾗ γὰρ διαιρούμενον ὁρᾶται εἰς ἄπειρον, ταύτῃ προστιθέμενον φανεῖται πρὸς τὸ ὡρισμένον. ἐν γὰρ τῷ πεπερασμένῳ μεγέθει ἂν λαβών τις ὡρισμένον προσλαμβάνῃ τῷ αὐτῷ λόγῳ, μὴ τὸ αὐτό τι τοῦ ὅλου μέγεθος περιλαμβάνων, οὐ διέξεισι τὸ πεπερασμένον· ἐὰν δ' οὕτως αὔξῃ τὸν λόγον ὥστε ἀεί τι

---

[a] After criticizing the beliefs of the Pythagoreans and Plato's school, Aristotle has just shown that there cannot be an infinite sensible body.

[b] The doctrine of "indivisible lines" is attributed to Plato by Aristot. *Met.* 992 a 20-22 and to Xenocrates, who succeeded Speusippus as head of the Academy, by Proclus

(*b*) The Infinite[a]

Aristotle, *Physics* Γ 6, 206 a 9-18

But it is clear that the complete denial of an infinite leads to many impossibilities. Time will have a beginning and an end, there will be magnitudes not divisible into magnitudes, and number will not be infinite. Since neither of these opposing views can be accepted, there is need of an arbitrator, and clearly each view must be in some sense true, in some sense untrue. Now " to be " is used in the sense either to exist actually or to exist potentially, while what is infinite is infinite either by addition or by division. It has already been stated that spatial extension is not infinite in actuality, but it is so by division ; for it is not difficult to refute the belief in indivisible lines[b] ; therefore it follows that the spatially infinite exists potentially.

Ibid. Γ 6, 206 b 3-12

The infinite in respect of addition is in a sense the same as the infinite in respect of division, the process of addition in a finite magnitude taking place conversely to that of division ; but where division is seen to go on *ad infinitum*, the converse process of addition tends to a definite limit. For if in a finite magnitude you take a determinate part and add to it in the same ratio, provided the successive added terms are not of the same magnitude, you will not come to the end of the finite magnitude ; but if the ratio is increased so that the terms added are always of the same

in Tim. 36 B, ed. Diehl ii. 246. and *in Eucl.* i., ed. Friedlein 279. 5, as well as by the commentators on Aristotle. The pseudo-Aristotelian tract *De lineis insecabilibus* seems directed against Xenocrates.

τὸ αὐτὸ περιλαμβάνειν μέγεθος, διέξεισι, διὰ τὸ πᾶν πεπερασμένον ἀναιρεῖσθαι ὁτῳοῦν ὡρισμένῳ.

Ibid. Γ 6, 206 b 27–207 a 7

Πλάτων διὰ τοῦτο δύο τὰ ἄπειρα ἐποίησεν, ὅτι καὶ ἐπὶ τὴν αὔξην δοκεῖ ὑπερβάλλειν καὶ εἰς ἄπειρον ἰέναι καὶ ἐπὶ τὴν καθαίρεσιν. ποιήσας μέντοι δύο οὐ χρῆται· οὔτε γὰρ ἐν τοῖς ἀριθμοῖς τὸ ἐπὶ τὴν καθαίρεσιν ἄπειρον ὑπάρχει (ἡ γὰρ μονὰς ἐλάχιστον), οὔτε τὸ ἐπὶ τὴν αὔξην (μέχρι γὰρ δεκάδος ποιεῖ τὸν ἀριθμόν).

Συμβαίνει δὲ τοὐναντίον εἶναι ἄπειρον ἢ ὡς λέγουσιν. οὐ γὰρ οὗ μηδὲν ἔξω, ἀλλ' οὗ ἀεί τι ἔξω ἐστί, τοῦτο ἄπειρόν ἐστιν. σημεῖον δέ· καὶ γὰρ τοὺς δακτυλίους ἀπείρους λέγουσι τοὺς μὴ ἔχοντας σφενδόνην, ὅτι αἰεί τι ἔξω ἔστι λαμβάνειν,

---

[a] From a finite magnitude AA′ a "determinate part" (ὡρισμένον) AB is cut off. BA′ is then divided at C, CA′ at

D and so on, in such a manner that the fractions diminish in the same ratio, *i.e.*, AB, BC, CD . . . form a geometrical progression. If the fractions diminish in this way, then AA′ will never be exhausted by this process, which will proceed *ad infinitum*. We may then look on AA′ as divided into an infinite number of parts, giving an *infinite by division*, or we may look on AB as having added to it an infinite number of parts, giving an *infinite by addition*. But if the successive added fractions are equal to each other, *i.e.* AB = BC = CD = . . ., then AA′ will be exhausted in a finite number of steps. This statement is equivalent to the

magnitude, you will come to the end, since any finite magnitude is exhausted by continually subtracting from it any definite fraction whatsoever.[a]

Ibid. Γ 6, 206 b 27–207 a 7

Plato posited two infinites [b] for this reason, that it is possible to proceed without limit both by way of increase and by way of diminution. But although he posits two infinites he does not use them; for in numbers there is for him no infinite by way of diminution (for the unit is a minimum), nor by way of increase (for he makes number go up to ten).[c]

So it comes about that the infinite is the opposite of what it is usually said to be. Not that beyond which there is nothing, but that of which there is always something beyond, is infinite. An illustration is given by the rings not having a bezel which are called endless, because there is always something beyond any

Axiom of Archimedes, already used by Eudoxus (see *supra*, p. 319 n. *b*).

[b] The reference is evidently to the famous "undetermined dyad of the great and small." A. E. Taylor (*Mind*, xxxv., pp. 419-440, 1926, and xxxvi., pp. 12-33, 1927) puts forward an ingenious theory of the nature of the "undetermined dyad." He sees a reference to the process of approximating more and more closely to a number by approximations alternately greater and less; D'Arcy Wentworth Thompson (*Mind*, xxxviii., pp. 43-55, 1929) adds the further refinement that the method is approximation by continued fractions. Though such conceptions were doubtless not beyond the mathematical capacity of Plato's Academy, they must remain guesses; and there is nothing to force us to believe that there is anything more profound in the concept of the undetermined dyad than Aristotle here indicates, viz., it is possible to proceed in an infinite series either by way of increase or by way of diminution.

[c] Aristotle has probably misunderstood some *obiter dictum* of Plato's.

καθ' ὁμοιότητα μέν τινα λέγοντες, οὐ μέντοι κυρίως· δεῖ γὰρ τοῦτό τε ὑπάρχειν καὶ μηδέ ποτε τὸ αὐτὸ λαμβάνεσθαι· ἐν δὲ τῷ κύκλῳ οὐ γίγνεται οὕτως, ἀλλ' αἰεὶ τὸ ἐφεξῆς μόνον ἕτερον.

Ibid. Γ 7, 207 b 27-34

Οὐκ ἀφαιρεῖται δ' ὁ λόγος οὐδὲ τοὺς μαθηματικοὺς τὴν θεωρίαν, ἀναιρῶν οὕτως εἶναι ἄπειρον ὥστε ἐνεργείᾳ εἶναι ἐπὶ τὴν αὔξησιν ἀδιεξίτητον· οὐδὲ γὰρ νῦν δέονται τοῦ ἀπείρου (οὐ γὰρ χρῶνται), ἀλλὰ μόνον εἶναι ὅσην ἂν βούλωνται πεπερασμένην· τῷ δὲ μεγίστῳ μεγέθει τὸν αὐτὸν ἔστι τετμῆσθαι λόγον ὁπηλικονοῦν μέγεθος ἕτερον. ὥστε πρὸς μὲν τὸ δεῖξαι ἐκείνοις οὐδὲν διοίσει τὸ εἶναι ἐν τοῖς οὖσιν μεγέθεσιν.

(c) Proof differing from Euclid's

Aristot. *Anal. Pr.* i. 24, 41 b 5-22

Μᾶλλον δὲ γίνεται φανερὸν ἐν τοῖς διαγράμμασιν, οἷον ὅτι τοῦ ἰσοσκελοῦς ἴσαι αἱ πρὸς τῇ βάσει. ἔστωσαν εἰς τὸ κέντρον ἠγμέναι αἱ ΑΒ. εἰ οὖν ἴσην λαμβάνοι τὴν ΑΓ γωνίαν τῇ ΒΔ μὴ ὅλως

[a] Aristotle had been arguing that in any syllogism one of the propositions must be affirmative and universal.

[b] Lit. "drawn."

[c] For this method of expressing the sum of two angles by the juxtaposition of the letters representing them, see Archytas's method of representing the sum of two numbers

point on them, but they are so called only after a certain resemblance, and not strictly; for this ought to be an essential attribute, and the same point should never do duty again; but in the circle this is not so, but the same point is used over and over.

Ibid. Γ 7, 207 b 27-34

But the argument does not deprive mathematicians of their study, although it denies that the infinite exists in the sense of actual existence as something increased to such an extent that it cannot be gone through; for even as it is they do not need the infinite (or use it), but only require that the finite straight line shall be as long as they please. Now any other magnitude may be divided in the same ratio as the largest magnitude. Hence it will make no difference to them, for the purpose of demonstration, whether there is actually an infinite among existing magnitudes.

(c) Proof differing from Euclid's

Aristotle, *Prior Analytics* i. 24, 41 b 5-22

This[a] is made clearer by geometrical theorems, such as that the angles at the base of an isosceles triangle are equal [Eucl. i. 5]. For let A, B be joined[b] to the centre. If then we assumed that the angle ΑΓ [*i.e.* Α + Γ][c] is equal to the angle ΒΔ [*i.e.* Β + Δ]

supra, p. 130. The angles A, B are the angles OAB, OBA, and are the same as those later described, in a confusing manner, as E, Z. The angles Γ, Δ are the smaller angles between AB and the *arc* of the circle. There is other evidence that such "mixed" angles played a big part in pre-Euclidean geometry, but Euclid himself scarcely uses them.

ἀξιώσας ἴσας τὰς τῶν ἡμικυκλίων, καὶ πάλιν τὴν Γ τῇ Δ μὴ πᾶσαν προσλαβὼν τὴν τοῦ τμήματος, ἔτι δ' ἀπ' ἴσων οὐσῶν τῶν ὅλων γωνιῶν καὶ ἴσων ἀφῃρημένων ἴσας εἶναι τὰς λοιπὰς τὰς ΕΖ, τὸ ἐξ ἀρχῆς αἰτήσεται, ἐὰν μὴ λάβῃ ἀπὸ τῶν ἴσων ἴσων ἀφαιρουμένων ἴσα λείπεσθαι.

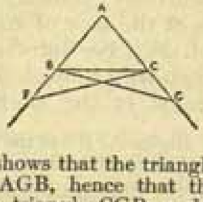
(d) Mechanics

(i.) *Principle of the Lever*

[Aristot.] *Mech.* 3, 850 a–b

Ἐπεὶ δὲ θᾶττον ὑπὸ τοῦ ἴσου βάρους κινεῖται ἡ[b] μείζων τῶν ἐκ τοῦ κέντρου, ἔστι δὲ τρία τὰ περὶ τὸν μοχλόν, τὸ μὲν ὑπομόχλιον, σπάρτον καὶ κέντρον, δύο δὲ βάρη, ὅ τε κινῶν καὶ τὸ κινούμενον· ὃ οὖν τὸ κινούμενον βάρος πρὸς τὸ κινοῦν, τὸ μῆκος πρὸς τὸ μῆκος ἀντιπέπονθεν. αἰεὶ δὲ ὅσῳ ἂν μεῖζον ἀφεστήκῃ τοῦ ὑπομοχλίου, ῥᾷον κινήσει. αἰτία δέ ἐστιν ἡ προλεχθεῖσα, ὅτι ἡ πλεῖον ἀπ-

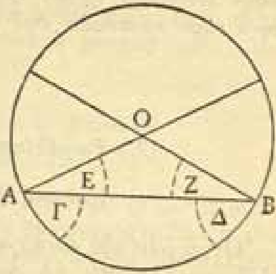
[a] Euclid proves this theorem by producing the equal sides AB, AC of an isosceles triangle to F, G where AF is



equal to AG. He shows that the triangle AFC is congruent with the triangle AGB, hence that the triangle BFC is congruent with the triangle CGB, and so finally that the angle ABC is equal to the angle ACB.

[b] The *Mechanics* is not by Aristotle, but must have been

without asserting generally that the angles of semicircles are equal, and again that the angle Γ is equal to the angle Δ without assuming generally that the two angles of all segments are equal, and if we further inferred that, since the whole angles are equal, and equal angles have been subtracted from them, the remaining angles E, Z are equal, we should commit a *petitio principii* unless we assumed generally that if equals are subtracted from equals the remainders are equal.[a]



(*d*) Mechanics

(i.) *Principle of the Lever*

[Aristotle], *Mechanics* 3, 850 a–b [b]

Since the greater radius is moved more quickly than the less by an equal weight, and there are three elements in the lever, the fulcrum, that is the cord[c] or centre, and two weights, that which moves and that which is moved, therefore the ratio of the weight moved to the moving weight is the inverse ratio of their distances from the fulcrum. It is always true that the farther the moving weight is away from the fulcrum, the more easily will it move. The reason is

written by someone under his influence at a not much later date; it may be taken as reflecting Aristotle's own ideas.

[c] The author has compared the fulcrum supporting a lever to the cord by which the beam of a balance is suspended.

ἔχουσα ἐκ τοῦ κέντρου μείζονα κύκλον γράφει. ὥστε ἀπὸ τῆς αὐτῆς ἰσχύος πλέον μεταστήσεται τὸ κινοῦν τὸ πλεῖον τοῦ ὑπομοχλίου ἄπεχον.

(ii.) *Parallelogram of Velocities*

[Aristot.] *Mech.* 1, 848 b

Ὅταν μὲν οὖν ἐν λόγῳ τινὶ φέρηται, ἐπ' εὐθείας ἀνάγκη φέρεσθαι τὸ φερόμενον, καὶ γίνεται διάμετρος αὐτὴ τοῦ σχήματος ὃ ποιοῦσιν αἱ ἐν τούτῳ τῷ λόγῳ συντεθεῖσαι γραμμαί.

Ἔστω γὰρ ὁ λόγος ὃν φέρεται τὸ φερόμενον, ὃν ἔχει ἡ ΑΒ πρὸς τὴν ΑΓ· καὶ τὸ μὲν ΑΓ φερέσθω πρὸς τὸ Β, ἡ δὲ ΑΒ ὑποφερέσθω πρὸς τὴν ΗΓ· ἐνηνέχθω δὲ τὸ μὲν Α πρὸς τὸ Δ, ἡ δὲ ἐφ' ᾗ ΑΒ πρὸς τὸ Ε. εἰ οὖν ἐπὶ τῆς φορᾶς ὁ λόγος ἦν ὃν ἡ ΑΒ ἔχει πρὸς τὴν ΑΓ, ἀνάγκη καὶ τὴν ΑΔ πρὸς τὴν ΑΕ τοῦτον ἔχειν τὸν λόγον. ὅμοιον ἄρα ἐστὶ τῷ λόγῳ τὸ μικρὸν τετράπλευρον τῷ μείζονι, ὥστε καὶ ἡ αὐτὴ διάμετρος αὐτῶν, καὶ τὸ Α ἔσται πρὸς Ζ. τὸν αὐτὸν δὴ τρόπον δειχθήσεται κἂν ὁπουοῦν διαληφθῇ ἡ φορά· αἰεὶ γὰρ ἔσται ἐπὶ τῆς διαμέτρου. φανερὸν οὖν ὅτι τὸ κατὰ τὴν διάμετρον φερόμενον ἐν δύο φοραῖς ἀνάγκη τὸν τῶν πλευρῶν φέρεσθαι λόγον.

---

[a] *i.e.* has two linear movements in a constant ratio to each other.

[b] *i.e.* parallelogram.

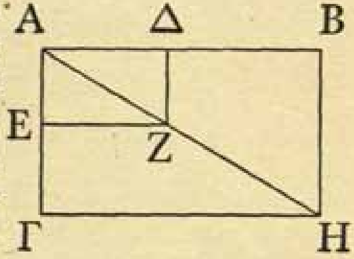
that already stated, that the point which is farther from the centre describes the greater circle. As a result, if the power applied is the same, that which moves the system will have a greater effect the farther it is from the fulcrum.

(ii.) *Parallelogram of Velocities*

[Aristotle], *Mechanics* 1, 848 b

When a body is moved in a certain ratio,[a] it must move in a straight line, and this straight line is the diagonal of the figure [b] formed from the two straight lines which have the given ratio.

For let the ratio according to which the body moves be that of ΑΒ to ΑΓ; let ΑΓ be moved towards Β while ΑΒ be moved towards ΗΓ; and let Α travel to Δ, while ΑΒ travels to a position marked by Ε. If the ratio of the movement is that of ΑΒ to ΑΓ, then ΑΔ must needs have the same ratio to ΑΕ. Therefore the small quadrilateral is similar to the larger, so that they have the same diagonal, and Α will be at Ζ. It may be shown that it will behave in the same manner wherever the motion be interrupted; it will be always on the diagonal. Therefore it is also manifest that a body travelling along the diagonal with two movements will travel according to the ratio of the sides.



XV. EUCLID

XV. EUCLID

(a) General

Stob. *Ecl.* ii. 31. 114, ed. Wachsmuth ii. 228. 25-29

Παρ' Εὐκλείδῃ τις ἀρξάμενος γεωμετρεῖν, ὡς τὸ πρῶτον θεώρημα ἔμαθεν, ἤρετο τὸν Εὐκλείδην· "τί δέ μοι πλέον ἔσται ταῦτα μαθόντι;" καὶ ὁ Εὐκλείδης τὸν παῖδα καλέσας "δός," ἔφη, "αὐτῷ τριώβολον, ἐπειδὴ δεῖ αὐτῷ ἐξ ὧν μανθάνει κερδαίνειν."

(b) The Elements

(i.) *Foundations*

Eucl. *Elem.* i.

Ὅροι

α'. Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.
β'. Γραμμὴ δὲ μῆκος ἀπλατές.
γ'. Γραμμῆς δὲ πέρατα σημεῖα.

---

[a] Hardly anything is known of the life of Euclid beyond what has already been stated in the passage quoted from Proclus (*supra*, p. 154). From Pappus vii. 35, ed. Hultsch ii. 678. 10-12, *infra*, p. 489, we infer the additional detail that he taught at Alexandria and founded a school there. Arabian references are summarized by Heath, *The Thirteen Books of Euclid's Elements*, 2nd edn., 1926, vol. i. pp. 4-6. Euclid must have flourished *c.* 300 B.C.

XV. EUCLID[a]

(a) General

Stobaeus, *Extracts* ii. 31. 114, ed. Wachsmuth ii. 228. 25-29

Someone who had begun to read geometry with Euclid, when he had learnt the first theorem asked Euclid, "But what advantage shall I get by learning these things?" Euclid called his slave and said, "Give him threepence, since he must needs make profit out of what he learns."

(b) The Elements[b]

(i.) *Foundations*

Euclid, *Elements* i.

DEFINITIONS[c]

1. A *point* is that which has no part.
2. A *line* is length without breadth.
3. The extremities of a line are points.

[b] For the meaning of *elements*, see *supra*, p. 150 n. *c*.

[c] For a full discussion of the many problems raised by Euclid's definitions, postulates and common notions the reader is referred to Heath, *The Thirteen Books of Euclid's Elements*, vol. i. pp. 155-240.

δ΄. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται.

ε΄. Ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.

ς΄. Ἐπιφανείας δὲ πέρατα γραμμαί.

ζ΄. Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται.

η΄. Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ' εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.

θ΄. Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία.

ι΄. Ὅταν δὲ εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστι, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ' ἣν ἐφέστηκεν.

ια΄. Ἀμβλεῖα γωνία ἐστὶν ἡ μείζων ὀρθῆς.

ιβ΄. Ὀξεῖα δὲ ἡ ἐλάσσων ὀρθῆς.

ιγ΄. Ὅρος ἐστίν, ὅ τινός ἐστι πέρας.

ιδ΄. Σχῆμά ἐστι τὸ ὑπό τινος ἤ τινων ὅρων περιεχόμενον.

ιε΄. Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἣ καλεῖται περιφέρεια], πρὸς ἣν ἀφ' ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.

ις΄. Κέντρον δὲ τοῦ κύκλου τὸ σημεῖον καλεῖται.

ιζ΄. Διάμετρος δὲ τοῦ κύκλου ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ'

---

[a] Plato (*Parmenides* 137 E) defines a straight line as " that of which the middle covers the ends." Euclid appears to be trying to say the same kind of thing in more geometrical

4. A *straight line* is a line which lies evenly with the points on itself.[a]

5. A *surface* is that which has length and breadth only.

6. The extremities of a surface are lines.

7. A *plane surface* is a surface which lies evenly with the straight lines on itself.

8. A *plane angle* is the inclination towards one another of two lines in a plane which meet one another and do not lie in a straight line.

9. And when the lines containing the angle are straight, the angle is called *rectilineal*.

10. When a straight line set up on a straight line makes the adjacent angles equal one to another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands.

11. An *obtuse angle* is an angle greater than a right angle.

12. An *acute angle* is an angle less than a right angle.

13. A *boundary* is that which is the extremity of anything.

14. A *figure* is that which is contained by any boundary or boundaries.

15. A *circle* is a plane figure contained by one line such that all the straight lines falling on it from one point among those lying within the figure are equal one to another.

16. And the point is called the *centre* of the circle.

17. A *diameter* of the circle is any straight line drawn through the centre and terminated in both

language. Neither statement is satisfactory as a definition (*cf.* Def. 7).

ἑκάτερα τὰ μέρη ὑπὸ τῆς τοῦ κύκλου περιφερείας, ἥτις καὶ δίχα τέμνει τὸν κύκλον.

ιη'. Ἡμικύκλιον δέ ἐστι τὸ περιεχόμενον σχῆμα ὑπό τε τῆς διαμέτρου καὶ τῆς ἀπολαμβανομένης ὑπ' αὐτῆς περιφερείας. κέντρον δὲ τοῦ ἡμικυκλίου τὸ αὐτό, ὃ καὶ τοῦ κύκλου ἐστίν.

ιθ'. Σχήματα εὐθύγραμμά ἐστι τὰ ὑπὸ εὐθειῶν περιεχόμενα, τρίπλευρα μὲν τὰ ὑπὸ τριῶν, τετράπλευρα δὲ τὰ ὑπὸ τεσσάρων, πολύπλευρα δὲ τὰ ὑπὸ πλειόνων ἢ τεσσάρων εὐθειῶν περιεχόμενα.

κ'. Τῶν δὲ τριπλεύρων σχημάτων ἰσόπλευρον μὲν τρίγωνόν ἐστι τὸ τὰς τρεῖς ἴσας ἔχον πλευράς, ἰσοσκελὲς δὲ τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς, σκαληνὸν δὲ τὸ τὰς τρεῖς ἀνίσους ἔχον πλευράς.

κα'. Ἔτι δὲ τῶν τριπλεύρων σχημάτων ὀρθογώνιον μὲν τρίγωνόν ἐστι τὸ ἔχον ὀρθὴν γωνίαν, ἀμβλυγώνιον δὲ τὸ ἔχον ἀμβλεῖαν γωνίαν, ὀξυγώνιον δὲ τὸ τὰς τρεῖς ὀξείας ἔχον γωνίας.

κβ'. Τῶν δὲ τετραπλεύρων σχημάτων τετράγωνον μέν ἐστιν, ὃ ἰσόπλευρόν τέ ἐστι καὶ ὀρθογώνιον, ἑτερόμηκες δέ, ὃ ὀρθογώνιον μέν, οὐκ ἰσόπλευρον δέ, ῥόμβος δέ, ὃ ἰσόπλευρον μέν, οὐκ ὀρθογώνιον δέ, ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευράς τε καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οὔτε ἰσόπλευρόν ἐστιν οὔτε ὀρθογώνιον· τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλείσθω.

κγ'. Παράλληλοί εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ' ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

---

[a] Heath classifies modern definitions of parallel straight

directions by the circumference of the circle, and such a straight line bisects the circle.

18. A *semicircle* is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.

19. *Rectilineal figures* are those contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.

20. Of trilateral figures an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has only two of its sides equal, and a *scalene triangle* that which has its three sides unequal.

21. Further, of trilateral figures, a *right-angled triangle* is that which has a right angle, an *obtuse-angled triangle* is that which has an obtuse angle, and an *acute-angled triangle* is that which has its three angles acute.

22. Of quadrilateral figures, a *square* is that which is both equilateral and right-angled; an *oblong* is that which is right-angled but not equilateral; a *rhombus* is that which is equilateral but not right-angled; and a *rhomboid* is that which has its opposite sides and angles equal one to another but is neither equilateral nor right-angled; and let quadrilaterals other than these be called *trapezia*.

23. *Parallel* straight lines are straight lines which, being in the same plane and produced indefinitely in both directions, do not meet one another in either direction.[a]

lines into three main groups: (1) *Parallel straight lines have no point common*, under which general conception the following varieties of statement are included: (a) *they do*

Αἰτήματα

α'. Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β'. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ' εὐθείας ἐκβαλεῖν.

γ'. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

δ'. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

ε'. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ' ἄπειρον συμπίπτειν, ἐφ' ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

---

not cut one another, (b) *they meet at infinity*, (c) *they have a common point at infinity*; (2) *parallel straight lines have the same, or like, direction or directions*; (3) *parallel straight lines have the distance between them constant*. Euclid's definition belongs to 1(a), and he avoids many fallacies latent in the other definitions, showing himself superior not only to many ancient, but to many modern, geometers.

[a] The chief purpose of these first three postulates is perhaps not to lay down that straight lines and circles can be drawn, but to delineate the nature of Euclidean space. They imply that space is continuous (not discrete) and infinite (not limited).

[b] This gives a determinate magnitude by which angles

POSTULATES

1. Let the following be postulated: to draw a straight line from any point to any point.

2. To produce a finite straight line continuously in a straight line.

3. To describe a circle with any centre and diameter.[a]

4. All right angles are equal one to another.[b]

5. If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.[c]

can be measured, but it does far more. To prove this statement it would be necessary to assume the *invariability of figures*. Euclid preferred to postulate the equality of right angles, which amounts to an assumption of the *invariability of figures* or the *homogeneity of space*.

[c] Heath says that this postulate "must ever be regarded as among the most epoch-making achievements in the domain of geometry," and observes: "When we consider the countless successive attempts made through more than twenty centuries to prove the postulate, many of them by geometers of ability, we cannot but admire the genius of the man who concluded that such a hypothesis, which he found necessary to the validity of his whole system of geometry, was really indemonstrable."

The postulate was frequently attacked in antiquity and many attempts have been made to prove it—by Ptolemy and Proclus in ancient days, by Wallis, Saccheri, Lambert and Legendre in modern times. All have failed. By omitting this postulate, Lobachewsky, Bolyai and Riemann developed "non-Euclidean" systems of geometry. Saccheri, in his book *Euclides ab omni naevo vindicatus* (1733), saw the possibility of alternative hypotheses, and worked out the consequences of several; but his faith in Euclidean geometry as the sole possible geometry was so strong that he failed to realize the full implications of his work.

Κοιναὶ ἔννοιαι

α′. Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.

β′. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.

γ′. Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενά ἐστιν ἴσα.

[δ′. Καὶ ἐὰν ἀνίσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἄνισα.

ε′. Καὶ τὰ τοῦ αὐτοῦ διπλάσια ἴσα ἀλλήλοις ἐστίν.

ϛ′. Καὶ τὰ τοῦ αὐτοῦ ἡμίση ἴσα ἀλλήλοις ἐστίν.]

ζ′. Καὶ τὰ ἐφαρμόζοντα ἐπ' ἄλληλα ἴσα ἀλλήλοις ἐστίν.

η′. Καὶ τὸ ὅλον τοῦ μέρους μεῖζόν [ἐστιν].

[θ′. Καὶ δύο εὐθεῖαι χωρίον οὐ περιέχουσιν.]

(ii.) *Theory of Proportion*

Eucl. *Elem.* v.

Ὅροι

α′. Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἔλασσον τοῦ μείζονος, ὅταν καταμετρῇ τὸ μεῖζον.

β′. Πολλαπλάσιον δὲ τὸ μεῖζον τοῦ ἐλάττονος, ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάττονος.

γ′. Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητά ποια σχέσις.

δ′. Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

ε′. Ἐν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον,

COMMON NOTIONS

1. Things which are equal to the same thing are equal one to another.

2. If equals are added to equals, the wholes are equal.

3. If equals are subtracted from equals, the remainders are equal.

7. Things which coincide with one another are equal one to another.

8. The whole is greater than the part.[a]

(ii.) *Theory of Proportion*

Euclid, *Elements* v.

DEFINITIONS

1. A magnitude is a *part* of a magnitude, the less of the greater, when it measures the greater.

2. The greater is a multiple of the less when it is measured by the less.

3. A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind.

4. Magnitudes are said to have a ratio one to another which are capable, when multiplied, of exceeding one another.

5. Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when,

[a] The MSS. have four other Common Notions, but they are unnecessary, and their genuineness was suspected even in antiquity. They are: 4. If equals are added to unequals, the wholes are unequal; 5. Things which are double of the same thing are equal one to another; 6. Things which are halves of the same thing are equal one to another; 9. Two straight lines do not enclose a space.

ὅταν τὰ τοῦ πρώτου καὶ τρίτου ἰσάκις πολλαπλάσια τῶν τοῦ δευτέρου καὶ τετάρτου ἰσάκις πολλαπλασίων καθ' ὁποιονοῦν πολλαπλασιασμὸν ἑκάτερον ἑκατέρου ἢ ἅμα ὑπερέχῃ ἢ ἅμα ἴσα ᾖ ἢ ἅμα ἐλλείπῃ ληφθέντα κατάλληλα.

ϛʹ. Τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη ἀνάλογον καλείσθω.

ζʹ. Ὅταν δὲ τῶν ἰσάκις πολλαπλασίων τὸ μὲν τοῦ πρώτου πολλαπλάσιον ὑπερέχῃ τοῦ τοῦ δευτέρου πολλαπλασίου, τὸ δὲ τοῦ τρίτου πολλαπλάσιον μὴ ὑπερέχῃ τοῦ τοῦ τετάρτου πολλαπλασίου, τότε τὸ πρῶτον πρὸς τὸ δεύτερον μείζονα λόγον ἔχειν λέγεται, ἤπερ τὸ τρίτον πρὸς τὸ τέταρτον.

ηʹ. Ἀναλογία δὲ ἐν τρισὶν ὅροις ἐλαχίστη ἐστίν.

θʹ. Ὅταν δὲ τρία μεγέθη ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τρίτον διπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον.

---

[a] In the translation of this remarkable definition I cannot improve on Heath. Literal translation is difficult because the words καθ' ὁποιονοῦν πολλαπλασιασμόν come only once in the Greek but refer both to τὰ . . . ἰσάκις πολλαπλάσια in the nominative and τῶν . . . ἰσάκις πολλαπλασίων in the genitive.

The definition, which avoids all mention of a part of a magnitude (unlike *Elements* vii. Def. 21), is applicable to all magnitudes, commensurable and incommensurable. It must be due, in substance at least, to Eudoxus (see *supra*, p. 408). The definition has often been assailed through misunderstanding, but has been brilliantly defended by such great mathematicians as Barrow and De Morgan, and was adopted by Weierstrass for his definition of equal numbers.

if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.[a]

6. Let magnitudes which have the same ratio be called *proportional*.

7. When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said *to have a greater ratio* to the second than the third has to the fourth.

8. A proportion in three terms is the least possible.

9. When three magnitudes are proportional, the first is said to have to the third the *duplicate ratio* of that which it has to the second.[b]

[a] Max Simon (*Euclid und die sechs planimetrischen Bücher*, p. 110) thinks it is clear from this definition that the Greeks possessed a notion of number as general as modern mathematicians. Heath (*The Thirteen Books of Euclid's Elements*, ii., pp. 124-126) shows how Euclid's definition divides all rational numbers into two *coextensive* classes, and so defines equal ratios in a manner exactly corresponding to Dedekind's theory of the irrational.

De Morgan gives the following modern equivalent of the definition. "Four magnitudes, A and B of one kind, and C and D of the same or another kind, are proportional when all the multiples of A can be distributed among the multiples of B in the same intervals as the corresponding multiples of C among those of D." That is to say, m, n being any numbers whatsoever, if mA lies between nB and $(n+1)$B, mC lies between nD and $(n+1)$D.

[b] If $\frac{a}{x}=\frac{x}{b}$, then $\frac{a}{b}=\frac{a^2}{x^2}$, and a has to b the *duplicate* ratio of a to x.

ι'. Ὅταν δὲ τέσσαρα μεγέθη ἀνάλογον ᾖ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον, καὶ ἀεὶ ἑξῆς ὁμοίως, ὡς ἂν ἡ ἀναλογία ὑπάρχῃ.

ια'. Ὁμόλογα μεγέθη λέγεται τὰ μὲν ἡγούμενα τοῖς ἡγουμένοις τὰ δὲ ἑπόμενα τοῖς ἑπομένοις.

ιβ'. Ἐναλλὰξ λόγος ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὸ ἡγούμενον καὶ τοῦ ἑπομένου πρὸς τὸ ἑπόμενον.

ιγ'. Ἀνάπαλιν λόγος ἐστὶ λῆψις τοῦ ἑπομένου ὡς ἡγουμένου πρὸς τὸ ἡγούμενον ὡς ἑπόμενον.

ιδ'. Σύνθεσις λόγου ἐστὶ λῆψις τοῦ ἡγουμένου μετὰ τοῦ ἑπομένου ὡς ἑνὸς πρὸς αὐτὸ τὸ ἑπόμενον.

ιε'. Διαίρεσις λόγου ἐστὶ λῆψις τῆς ὑπεροχῆς, ᾗ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου, πρὸς αὐτὸ τὸ ἑπόμενον.

ιϛ'. Ἀναστροφὴ λόγου ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὴν ὑπεροχήν, ᾗ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου.

ιζ'. Δι' ἴσου λόγος ἐστὶ πλειόνων ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος σύνδυο λαμβανομένων καὶ ἐν τῷ αὐτῷ λόγῳ, ὅταν ᾖ ὡς ἐν τοῖς πρώτοις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον, οὕτως ἐν τοῖς δευτέροις μεγέθεσι τὸ

---

[a] The magnitudes must be in continuous proportion. If $\frac{a}{x}=\frac{x}{y}=\frac{y}{b}$, then $\frac{a}{b}=\frac{a^3}{x^3}$, and a has to b the *triplicate* ratio of a to x. Alternatively, a cube with side a has the same ratio to a cube with side x as a to b (see *supra*, p. 258 n. b).

10. When four magnitudes are proportional[a] the first is said to have to the fourth the *triplicate* ratio of that which it has to the second, and so on continually, whatever the proportion.

11. The term *corresponding magnitudes* is used of antecedents in relation to antecedents and of consequents in relation to consequents.[b]

12. *Alternate ratio* means taking the antecedent in relation to the antecedent, and the consequent in relation to the consequent.[c]

13. *Inverse ratio* means taking the consequent as antecedent in relation to the antecedent as consequent.[d]

14. *Composition of a ratio* means taking the antecedent together with the consequent as one in relation to the consequent by itself.[e]

15. *Separation of a ratio* means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself.[f]

16. *Conversion of a ratio* means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent.[g]

17. A ratio *ex aequali* arises when, there being several magnitudes and another set equal to them in multitude which taken two by two are in the same proportion, as the first is to the last in the first set of magnitudes, so is the first to the last in the second

[b] "Antecedents" are literally "leading terms," "consequents" the "following terms." In the ratio $a:b$, a is the antecedent, b the consequent.

[c] If $a:b::A:B$, then $a:A::b:B$.

[d] If $a:b::A:B$, then $b:a::B:A$.

[e] *i.e.* the transformation of the ratio $a:b$ into $a+b:b$.

[f] *i.e.* the transformation of the ratio $a:b$ into $a-b:b$.

[g] *i.e.* the transformation of the ratio $a:b$ into $a:a-b$.

πρῶτον πρὸς τὸ ἔσχατον· ἢ ἄλλως· λῆψις τῶν ἄκρων καθ' ὑπεξαίρεσιν τῶν μέσων.

ιη'. Τεταραγμένη δὲ ἀναλογία ἐστίν, ὅταν τριῶν ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος γίνηται ὡς μὲν ἐν τοῖς πρώτοις μεγέθεσιν ἡγούμενον πρὸς ἑπόμενον, οὕτως ἐν τοῖς δευτέροις μεγέθεσιν ἡγούμενον πρὸς ἑπόμενον, ὡς δὲ ἐν τοῖς πρώτοις μεγέθεσιν ἑπόμενον πρὸς ἄλλο τι, οὕτως ἐν τοῖς δευτέροις ἄλλο τι πρὸς ἡγούμενον.

(iii.) *Theory of Incommensurables*

Eucl. *Elem.* x.

Ὅροι

α'. Σύμμετρα μεγέθη λέγεται τὰ τῷ αὐτῷ μέτρῳ μετρούμενα, ἀσύμμετρα δέ, ὧν μηδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι.

β'. Εὐθεῖαι δυνάμει σύμμετροί εἰσιν, ὅταν τὰ ἀπ' αὐτῶν τετράγωνα τῷ αὐτῷ χωρίῳ μετρῆται, ἀσύμμετροι δέ, ὅταν τοῖς ἀπ' αὐτῶν τετραγώνοις μηδὲν ἐνδέχηται χωρίον κοινὸν μέτρον γενέσθαι.

γ'. Τούτων ὑποκειμένων δείκνυται, ὅτι τῇ προτεθείσῃ εὐθείᾳ ὑπάρχουσιν εὐθεῖαι πλήθει ἄπειροι σύμμετροί τε καὶ ἀσύμμετροι αἱ μὲν μήκει μόνον, αἱ δὲ καὶ δυνάμει. καλείσθω οὖν ἡ μὲν προτεθεῖσα εὐθεῖα ῥητή, καὶ αἱ ταύτῃ σύμμετροι εἴτε μήκει

---

[a] δι' ἴσου must mean "at an equal distance," *i.e.*, after an equal number of terms. If $a, b, c \ldots m, n$ is one set of magnitudes and $A, B, C \ldots M, N$ the other, and $a : b = A : B$, and so on, up to $m : n = M : N$, then $a : n = A : N$. This is proved in v. 22. The definition merely serves to gave a name to the inference.

set of magnitudes ; in other words, a taking of the extremes by removal of the intermediate terms.[a]

18. A *perturbed proportion* arises when, there being three magnitudes and another set equal to them in multitude, as antecedent is to consequent in the first magnitudes, so is antecedent to consequent in the second magnitudes, while as the consequent is to the other term in the first magnitudes, so is the other term to the antecedent in the second magnitudes.[b]

(iii.) *Theory of Incommensurables*

Euclid, *Elements* x.

DEFINITIONS

1. Those magnitudes are said to be *commensurable* which are measured by the same common measure, and those incommensurable which cannot have any common measure.

2. Straight lines are *commensurable in square*, when the squares on them are measured by the same area, and *incommensurable in square* when the squares on them cannot have any area as a common measure.

3. With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called *rational*, and those straight lines which are commensurable with it, whether in length

[b] If a, b, c and A, B, C are the two sets of magnitudes, and $a : b = \mathrm{B} : \mathrm{C}$, $b : c = \mathrm{A} : \mathrm{B}$ the proportion is said to be *perturbed*. It follows that $a : c = \mathrm{A} : \mathrm{C}$. This is a particular case of the inference δι' ἴσου and is proved in v. 23.

καὶ δυνάμει εἴτε δυνάμει μόνον ῥηταί, αἱ δὲ ταύτῃ ἀσύμμετροι ἄλογοι καλείσθωσαν.

δʹ. Καὶ τὸ μὲν ἀπὸ τῆς προτεθείσης εὐθείας τετράγωνον ῥητόν, καὶ τὰ τούτῳ σύμμετρα ῥητά, τὰ δὲ τούτῳ ἀσύμμετρα ἄλογα καλείσθω, καὶ αἱ δυνάμεναι αὐτὰ ἄλογοι, εἰ μὲν τετράγωνα εἴη, αὐταὶ αἱ πλευραί, εἰ δὲ ἕτερά τινα εὐθύγραμμα, αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφουσαι.

αʹ

Δύο μεγεθῶν ἀνίσων ἐκκειμένων, ἐὰν ἀπὸ τοῦ μείζονος ἀφαιρεθῇ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους.

Ἔστω δύο μεγέθη ἄνισα τὰ ΑΒ, Γ, ὧν μεῖζον τὸ ΑΒ· λέγω, ὅτι ἐὰν ἀπὸ τοῦ ΑΒ ἀφαιρεθῇ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ Γ μεγέθους.

Τὸ Γ γὰρ πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ

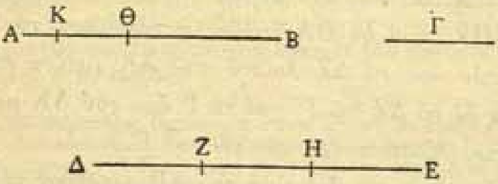
and in square or in square only, be called *rational*, but those which are incommensurable with it be called *irrational*.

4. And let the square on the assigned straight line be called *rational*, and those areas which are commensurable with it *rational*, but those which are incommensurable with it *irrational*, and the straight lines which produce them *irrational*, that is, if the areas are squares, the sides themselves, but if the areas are any other rectilineal figures, the straight lines on which are described squares equal to them.

Prop. 1

Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than the half, and from the remainder a magnitude greater than its half, and so on continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let AB, Γ be the two unequal magnitudes, of which AB is the greater ; I say that, if from AB there be



subtracted a magnitude greater than its half, and from the remainder a magnitude greater than its half, and so on continually, there will be left some magnitude which will be less than the magnitude Γ.

For Γ, if multiplied, will at some time be greater

ΑΒ μεῖζον. πεπολλαπλασιάσθω, καὶ ἔστω τὸ ΔΕ τοῦ μὲν Γ πολλαπλάσιον, τοῦ δὲ ΑΒ μεῖζον, καὶ διῃρήσθω τὸ ΔΕ εἰς τὰ τῷ Γ ἴσα τὰ ΔΖ, ΖΗ, ΗΕ, καὶ ἀφῃρήσθω ἀπὸ μὲν τοῦ ΑΒ μεῖζον ἢ τὸ ἥμισυ τὸ ΒΘ, ἀπὸ δὲ τοῦ ΑΘ μεῖζον ἢ τὸ ἥμισυ τὸ ΘΚ, καὶ τοῦτο ἀεὶ γιγνέσθω, ἕως ἂν αἱ ἐν τῷ ΑΒ διαιρέσεις ἰσοπληθεῖς γένωνται ταῖς ἐν τῷ ΔΕ διαιρέσεσιν.

Ἔστωσαν οὖν αἱ ΑΚ, ΚΘ, ΘΒ διαιρέσεις ἰσοπληθεῖς οὖσαι ταῖς ΔΖ, ΖΗ, ΗΕ· καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΔΕ τοῦ ΑΒ, καὶ ἀφῄρηται ἀπὸ μὲν τοῦ ΔΕ ἔλασσον τοῦ ἡμίσεως τὸ ΕΗ, ἀπὸ δὲ τοῦ ΑΒ μεῖζον ἢ τὸ ἥμισυ τὸ ΒΘ, λοιπὸν ἄρα τὸ ΗΔ λοιποῦ τοῦ ΘΑ μεῖζόν ἐστιν. καὶ ἐπεὶ μεῖζόν ἐστι τὸ ΗΔ τοῦ ΘΑ, καὶ ἀφῄρηται τοῦ μὲν ΗΔ ἥμισυ τὸ ΗΖ, τοῦ δὲ ΘΑ μεῖζον ἢ τὸ ἥμισυ τὸ ΘΚ, λοιπὸν ἄρα τὸ ΔΖ λοιποῦ τοῦ ΑΚ μεῖζόν ἐστιν. ἴσον δὲ τὸ ΔΖ τῷ Γ· καὶ τὸ Γ ἄρα τοῦ ΑΚ μεῖζόν ἐστιν. ἔλασσον ἄρα τὸ ΑΚ τοῦ Γ.

Καταλείπεται ἄρα ἀπὸ τοῦ ΑΒ μεγέθους τὸ ΑΚ μέγεθος ἔλασσον ὂν τοῦ ἐκκειμένου ἐλάσσονος μεγέθους τοῦ Γ· ὅπερ ἔδει δεῖξαι—ὁμοίως δὲ δειχθήσεται, κἂν ἡμίση ᾖ τὰ ἀφαιρούμενα.

than AB [see v. Def. 4]. Let it be multiplied, and let ΔE be a multiple of Γ, greater than AB, and let ΔE be divided into the parts ΔZ, ZH, HE equal to Γ, and from AB let there be subtracted BΘ greater than its half, and from AΘ let there be subtracted ΘK greater than its half, and so on continually, until the divisions in AB are equal in multitude to the divisions in ΔE.

Let, then, AK, KΘ, ΘB be divisions equal in multitude with ΔZ, ZH, HE; now since ΔE is greater than AB, and from ΔE there has been subtracted EH less than its half, and from AB there has been subtracted BΘ greater than its half, therefore the remainder HΔ is greater than the remainder ΘA. And since HΔ is greater than ΘA, and from HΔ there has been subtracted the half, HZ, and from ΘA there has been subtracted ΘK greater than its half, therefore the remainder ΔZ is greater than the remainder AK. Now ΔZ is equal to Γ; and therefore Γ is greater than AK. Therefore AK is less than Γ.

There is therefore left of the magnitude AB the magnitude AK which is less than the lesser magnitude set out, namely, Γ; which was to be proved—and this can be similarly proved even if the parts to be subtracted be halves.[a]

[a] This important theorem is often known as the Axiom of Archimedes because of the use to which he puts it, or a similar lemma: "The excess by which the greater of two unequal areas exceeds the lesser can, by being continually added to itself, be made to exceed any given finite area." Archimedes makes no claim to have discovered this lemma, which is doubtless due to Eudoxus. The chief use of the "axiom" by Euclid is to prove *Elements* xii. 2, that circles are to one another as the squares on their diameters.

Prop. 111, coroll.

Ἡ ἀποτομὴ καὶ αἱ μετ᾽ αὐτὴν ἄλογοι οὔτε τῇ μέσῃ οὔτε ἀλλήλαις εἰσὶν αἱ αὐταί. . . .

Καὶ ἐπεὶ δέδεικται ἡ ἀποτομὴ οὐκ οὖσα ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων, ποιοῦσι δὲ πλάτη παρὰ ῥητὴν

---

[a] Much of Eucl. *Elem.* x. is devoted to an elaborate classification of irrational straight lines. Zeuthen (*Geschichte der Mathematik im Altertum und Mittelalter*, p. 56) suggests that, inasmuch as one straight line looks very much like another, the Greeks could not perceive by simple inspection that difference among irrational quantities which our system of algebraic symbols enables us to see; consequently they were led to classify irrational straight lines in the manner of Eucl. *Elem.* x., and we know from an Arabic commentary on this book discovered by Woepcke (*Mémoires présentés à l'Académie des Sciences*, xiv., 1856, pp. 658-720) that Theaetetus had to some extent preceded Euclid. In this system irrational straight lines are classified according to the areas they produce when "applied" (*v. supra*, pp. 186-187) to other straight lines. For full details the reader must be referred to Loria, *Le scienze esatte nell' antica Grecia*, pp. 225-231, Heath's notes in *The Thirteen Books of Euclid's Elements*, vol. iii., and *H.G.M.* i. 404-411, but it may be useful to give here, in Heath's notation, the modern algebraic equivalents of Euclid's irrational straight lines. A medial line is of the form $k^{\frac{1}{4}}\rho$, *i.e.*, the positive solution of the equation $x^2 - \rho\sqrt{k}.\rho = 0$. The other twelve irrational lines are compound, and may best be arranged in pairs as follows:

1. $\left.\begin{array}{l}\textit{Binomial}\\ \textit{Apotome}\end{array}\right\}$ $\quad \rho \pm \sqrt{k}\,.\,\rho$,

being the positive roots of the equation

$$x^4 - 2(1+k)\rho^2\,.\,x^2 + (1-k)^2\rho^4 = 0.$$

2. $\left.\begin{array}{l}\textit{First bimedial}\\ \textit{First apotome of a medial}\end{array}\right\}$ $\quad k^{\frac{1}{4}}\rho \pm k^{\frac{3}{4}}\rho$,

being the positive roots of the equation

$$x^4 - 2\sqrt{k}(1+k)\rho^2\,.\,x^2 + k(1-k)^2\rho^4 = 0.$$

Prop. 111, corollary

The apotome and the irrational straight lines following it are the same neither with the medial straight line nor with one another.[a] . . .

Since the apotome has been proved not to be the same as the binomial straight line [x. 111], and, if

3. *Second bimedial*
Second apotome of a medial $\Big\}$ $k^{\frac{1}{4}}\rho \pm \dfrac{\lambda^{\frac{1}{2}}\rho}{k^{\frac{1}{4}}}$,

being the positive roots of the equation

$$x^4 - 2\frac{k+\lambda}{\sqrt{k}}\rho^2 \, . \, x^2 + \frac{(k-\lambda)^2}{k}\rho^4 = 0.$$

4. *Major*
Minor $\Big\}$ $\dfrac{\rho}{\sqrt{2}}\sqrt{\left(1+\dfrac{k}{\sqrt{1+k^2}}\right)} \pm \dfrac{\rho}{\sqrt{2}}\sqrt{\left(1-\dfrac{k}{\sqrt{1+k^2}}\right)}$,

being the positive roots of the equation

$$x^4 - 2\rho^2 \, . \, x^2 + \frac{k^2}{1+k^2}\rho^4 = 0.$$

5. *Side of a rational plus a medial area*
Producing with a rational area a medial whole $\Bigg\}$ $\dfrac{\rho}{\sqrt{2(1+k^2)}}\sqrt{(\sqrt{1+k^2}+k)} \pm \dfrac{\rho}{\sqrt{2(1+k^2)}}\sqrt{(\sqrt{1+k^2}-k)}$,

being the positive roots of the equation

$$x^4 - \frac{2}{\sqrt{(1+k^2)}}\rho^2 \, . \, x^2 + \frac{k^2}{(1+k^2)^2}\rho^4 = 0.$$

6. *Side of the sum of two medial areas*
Producing with a medial area a medial whole $\Bigg\}$ $\dfrac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{\left(1+\dfrac{k}{\sqrt{1+k^2}}\right)} \pm \dfrac{\rho\lambda^{\frac{1}{4}}}{\sqrt{2}}\sqrt{\left(1-\dfrac{k}{\sqrt{1+k^2}}\right)}$,

being the positive roots of the equation

$$x^4 - 2\sqrt{\lambda} \, . \, x^2\rho^2 + \lambda\frac{k^2}{1+k^2}\rho^4 = 0.$$

παραβαλλόμεναι αἱ μετὰ τὴν ἀποτομὴν ἀποτομὰς ἀκολούθως ἑκάστη τῇ τάξει τῇ καθ' αὑτήν, αἱ δὲ μετὰ τὴν ἐκ δύο ὀνομάτων τὰς ἐκ δύο ὀνομάτων καὶ αὗται τῇ τάξει ἀκολούθως, ἕτεραι ἄρα εἰσὶν αἱ μετὰ τὴν ἀποτομὴν καὶ ἕτεραι αἱ μετὰ τὴν ἐκ δύο ὀνομάτων, ὡς εἶναι τῇ τάξει πάσας ἀλόγους ιγ,

Μέσην,
Ἐκ δύο ὀνομάτων,
Ἐκ δύο μέσων πρώτην,
Ἐκ δύο μέσων δευτέραν,
Μείζονα,
Ῥητὸν καὶ μέσον δυναμένην,
Δύο μέσα δυναμένην,
Ἀποτομήν,
Μέσης ἀποτομὴν πρώτην,
Μέσης ἀποτομὴν δευτέραν,
Ἐλάσσονα,
Μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσαν,
Μετὰ μέσου μέσον τὸ ὅλον ποιοῦσαν.

(iv.) *Method of Exhaustion*

Eucl. *Elem.* xii. 2

Οἱ κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα.

Ἔστωσαν κύκλοι οἱ ΑΒΓΔ, ΕΖΗΘ, διάμετροι δὲ αὐτῶν αἱ ΒΔ, ΖΘ· λέγω, ὅτι ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ τετράγωνον.

[a] Eudemus attributed the discovery of this important theorem to Hippocrates (see *supra*, p. 238). Unfortunately we do not know how Hippocrates proved it.

applied to a rational straight line, the straight lines following the apotome produce, as breadths, apotomes according to their order, and those following the binomial straight line produce, as breadths, binomials according to their order, therefore the straight lines following the apotome are different, and the straight lines following the binomial straight line are different, so that in all there are, in order, thirteen straight lines,

Medial,
Binomial,
First bimedial,
Second bimedial,
Major,
Side of a rational plus a medial area,
Side of the sum of two medial areas,
Apotome,
First apotome of a medial straight line,
Second apotome of a medial straight line,
Minor,
Producing with a rational area a medial whole,
Producing with a medial area a medial whole.

(iv.) *Method of Exhaustion*

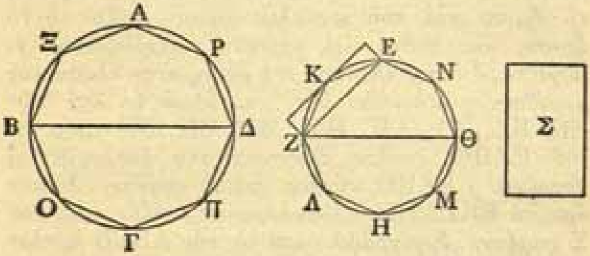
Euclid, *Elements* xii. 2 [a]

Circles are to one another as the squares on the diameters.

Let ΑΒΓΔ, ΕΖΗΘ be circles, and ΒΔ, ΖΘ their diameters ; I say that, as the circle ΑΒΓΔ is to the circle ΕΖΗΘ, so is the square on ΒΔ to the square on ΖΘ.

Εἰ γὰρ μή ἐστιν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ, οὕτως τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, ἔσται ὡς τὸ ἀπὸ τῆς ΒΔ πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος ἤτοι πρὸς ἔλασσόν τι τοῦ ΕΖΗΘ κύκλου χωρίον ἢ πρὸς μεῖζον. ἔστω πρότερον πρὸς ἔλασσον τὸ Σ. καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘ κύκλον τετράγωνον τὸ ΕΖΗΘ· τὸ δὴ ἐγγεγραμμένον τετράγωνον μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ ΕΖΗΘ κύκλου, ἐπειδήπερ ἐὰν διὰ τῶν Ε, Ζ, Η, Θ σημείων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν, τοῦ περιγραφομένου περὶ τὸν κύκλον τετραγώνου ἥμισύ ἐστι τὸ ΕΖΗΘ τετράγωνον, τοῦ δὲ περιγραφέντος τετραγώνου ἐλάττων ἐστὶν ὁ κύκλος· ὥστε τὸ ΕΖΗΘ ἐγγεγραμμένον τετράγωνον μεῖζόν ἐστι τοῦ ἡμίσεως τοῦ ΕΖΗΘ κύκλου. τετμήσθωσαν δίχα αἱ ΕΖ, ΖΗ, ΗΘ, ΘΕ περιφέρειαι κατὰ τὰ Κ, Λ, Μ, Ν σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΕΚ, ΚΖ, ΖΛ, ΛΗ, ΗΜ, ΜΘ, ΘΝ, ΝΕ· καὶ ἕκαστον ἄρα τῶν ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ τοῦ καθ' ἑαυτὸ τμήματος τοῦ κύκλου, ἐπειδήπερ ἐὰν διὰ τῶν Κ, Λ, Μ, Ν σημείων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν καὶ ἀναπληρώσωμεν τὰ ἐπὶ τῶν ΕΖ, ΖΗ, ΗΘ, ΘΕ εὐθειῶν παραλληλόγραμμα, ἕκαστον τῶν ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων ἥμισυ ἔσται τοῦ καθ' ἑαυτὸ παραλληλογράμμου, ἀλλὰ τὸ καθ'

For if the circle ΑΒΓΔ is not to the circle ΕΖΗΘ as the square on ΒΔ to the square on ΖΘ, then the square on ΒΔ will be to the square on ΖΘ as the circle ΑΒΓΔ is to some area either less than the circle ΕΖΗΘ or greater. Let it first be in that ratio to a lesser area Σ. And let the square ΕΖΗΘ be inscribed in the circle ΕΖΗΘ; then the inscribed square is greater than the half of the circle ΕΖΗΘ, inasmuch as, if through the points Ε, Ζ, Η, Θ we draw tangents to the circle, the square ΕΖΗΘ is half the square circumscribed about the circle, and the circle is less



than the circumscribed square; so that the inscribed square ΕΖΗΘ is greater than the half of the circle ΕΖΗΘ. Let the circumferences ΕΖ, ΖΗ, ΗΘ, ΘΕ be bisected at the points Κ, Λ, Μ, Ν, and let ΕΚ, ΚΖ, ΖΛ, ΛΗ, ΗΜ, ΜΘ, ΘΝ, ΝΕ be joined; therefore each of the triangles ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ is greater than the half of the segment of the circle about it, inasmuch as, if through the points Κ, Λ, Μ, Ν we draw tangents to the circle and complete the parallelograms on the straight lines ΕΖ, ΖΗ, ΗΘ, ΘΕ, each of the triangles ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ will be half of the parallelogram about it, while the segment

ἑαυτὸ τμῆμα ἔλαττόν ἐστι τοῦ παραλληλογράμμου· ὥστε ἕκαστον τῶν ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ τριγώνων μεῖζόν ἐστι τοῦ ἡμίσεως τοῦ καθ' ἑαυτὸ τμήματος τοῦ κύκλου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ τοῦτο ἀεὶ ποιοῦντες καταλείψομέν τινα ἀποτμήματα τοῦ κύκλου, ἃ ἔσται ἐλάσσονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ ΕΖΗΘ κύκλος τοῦ Σ χωρίου. ἐδείχθη γὰρ ἐν τῷ πρώτῳ θεωρήματι τοῦ δεκάτου βιβλίου, ὅτι δύο μεγεθῶν ἀνίσων ἐκκειμένων, ἐὰν ἀπὸ τοῦ μείζονος ἀφαιρεθῇ μεῖζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μεῖζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεταί τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους. λελείφθω οὖν, καὶ ἔστω τὰ ἐπὶ τῶν ΕΚ, ΚΖ, ΖΛ, ΛΗ, ΗΜ, ΜΘ, ΘΝ, ΝΕ τμήματα τοῦ ΕΖΗΘ κύκλου ἐλάττονα τῆς ὑπεροχῆς, ᾗ ὑπερέχει ὁ ΕΖΗΘ κύκλος τοῦ Σ χωρίου. λοιπὸν ἄρα τὸ ΕΚΖΛΗΜΘΝ πολύγωνον μεῖζόν ἐστι τοῦ Σ χωρίου. ἐγγεγράφθω καὶ εἰς τὸν ΑΒΓΔ κύκλον τῷ ΕΚΖΛΗΜΘΝ πολυγώνῳ ὅμοιον πολύγωνον τὸ ΑΞΒΟΓΠΔΡ· ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ τετράγωνον, οὕτως τὸ ΑΞΒΟΓΠΔΡ πολύγωνον πρὸς τὸ ΕΚΖΛΗΜΘΝ πολύγωνον. ἀλλὰ καὶ ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸ Σ χωρίον· καὶ ὡς ἄρα ὁ ΑΒΓΔ κύκλος πρὸς τὸ Σ χωρίον, οὕτως τὸ ΑΞΒΟΓΠΔΡ πολύγωνον πρὸς τὸ ΕΚΖΛΗΜΘΝ πολύγωνον· ἐναλλὰξ ἄρα ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸ ἐν αὐτῷ πολύγωνον, οὕτως τὸ Σ χωρίον πρὸς τὸ ΕΚΖΛΗΜΘΝ πολύγωνον. μείζων δὲ ὁ ΑΒΓΔ

about it is less than the parallelogram ; so that each of the triangles ΕΚΖ, ΖΛΗ, ΗΜΘ, ΘΝΕ is greater than the half of the segment of the circle about it. Thus, by bisecting the remaining circumferences and joining straight lines, and doing this continually, we shall leave some segments of the circle which will be less than the excess by which the circle ΕΖΗΘ exceeds the area Σ. For it was proved in the first theorem of the tenth book that, if two unequal magnitudes be set out, and if from the greater there be subtracted a magnitude greater than its half, and from the remainder a magnitude greater than its half, and so on continually, there will be left some magnitude which is less than the lesser magnitude set out. Let such segments be then left, and let the segments of the circle ΕΖΗΘ on ΕΚ, ΚΖ, ΖΛ, ΛΗ, ΗΜ, ΜΘ, ΘΝ, ΝΕ be less than the excess by which the circle ΕΖΗΘ exceeds the area Σ. Therefore the remainder, the polygon ΕΚΖΛΗΜΘΝ, is greater than the area Σ. Let there be inscribed, also, in the circle ΑΒΓΔ the polygon ΑΞΒΟΓΠΔΡ similar to the polygon ΕΚΖΛΗΜΘΝ ; therefore as the square on ΒΔ is to the square on ΖΘ, so is the polygon ΑΞΒΟΓΠΔΡ to the polygon ΕΚΖΛΗΜΘΝ [xii. 1]. But as the square on ΒΔ is to the square on ΖΘ, so is the circle ΑΒΓΔ to the area Σ ; therefore also as the circle ΑΒΓΔ is to the area Σ, so is the polygon ΑΞΒΟΓΠΔΡ to the polygon ΕΚΖΛΗΜΘΝ [v. 11] ; therefore, alternately, as the circle ΑΒΓΔ is to the polygon in it, so is the area Σ to the polygon ΕΚΖΛΗΜΘΝ. Now the circle

κύκλος τοῦ ἐν αὐτῷ πολυγώνου· μεῖζον ἄρα καὶ τὸ Σ χωρίον τοῦ ΕΚΖΛΗΜΘΝ πολυγώνου. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς ἔλασσόν τι τοῦ ΕΖΗΘ κύκλου χωρίον. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ὡς τὸ ἀπὸ ΖΘ πρὸς τὸ ἀπὸ ΒΔ, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλασσόν τι τοῦ ΑΒΓΔ κύκλου χωρίον.

Λέγω δή, ὅτι οὐδὲ ὡς τὸ ἀπὸ τῆς ΒΔ πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς μεῖζόν τι τοῦ ΕΖΗΘ κύκλου χωρίον.

Εἰ γὰρ δυνατόν, ἔστω πρὸς μεῖζον τὸ Σ. ἀνάπαλιν ἄρα ὡς τὸ ἀπὸ τῆς ΖΘ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΔΒ, οὕτως τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον. ἀλλ' ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΖΘ πρὸς τὸ ἀπὸ τῆς ΒΔ, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἔλασσόν τι τοῦ ΑΒΓΔ κύκλου χωρίον· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἐστὶν ὡς τὸ ἀπὸ ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς μεῖζόν τι τοῦ ΕΖΗΘ κύκλου χωρίον. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλασσον· ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς ΒΔ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΖΘ, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον.

Οἱ ἄρα κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα· ὅπερ ἔδει δεῖξαι.

ΑΒΓΔ is greater than the polygon in it; therefore the area Σ also is greater than the polygon ΕΚΖΛΗΜΘΝ. But it is also less; which is impossible. Therefore it is not true that, as is the square on ΒΔ to the square on ΖΘ, so is the circle ΑΒΓΔ to some area less than the circle ΕΖΗΘ. Similarly we shall prove that neither is it true that, as the square on ΖΘ is to the square on ΒΔ, so is the circle ΕΖΗΘ to some area less than the circle ΑΒΓΔ.

I say now that neither is the circle ΑΒΓΔ towards some area greater than the circle ΕΖΗΘ as the square on ΒΔ is to the square on ΖΘ.

For, if possible, let it be in that ratio to some greater area Σ. Therefore, inversely, as the square on ΖΘ is to the square on ΔΒ, so is the area Σ to the circle ΑΒΓΔ. But as the area Σ is to the circle ΑΒΓΔ, so is the circle ΕΖΗΘ to some area less than the circle ΑΒΓΔ; therefore also, as the square on ΖΘ is to the square on ΒΔ, so is the circle ΕΖΗΘ to some area less than the circle ΑΒΓΔ [v. 11]; which was proved impossible. Therefore it is not true that, as the square on ΒΔ is to the square on ΖΘ, so is the circle ΑΒΓΔ to some area greater than the circle ΕΖΗΘ. And it was proved not to be in that relation to a less area; therefore as the square on ΒΔ is to the square on ΖΘ, so is the circle ΑΒΓΔ to the circle ΕΖΗΘ.

Therefore circles are to one another as the squares on the diameters; which was to be proved.

(v.) *Regular Solids*

Eucl. *Elem.* xiii. 18

Τὰς πλευρὰς τῶν πέντε σχημάτων ἐκθέσθαι καὶ συγκρῖναι πρὸς ἀλλήλας.

Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ, καὶ τετμήσθω κατὰ τὸ Γ ὥστε ἴσην εἶναι τὴν ΑΓ τῇ ΓΒ, κατὰ δὲ τὸ Δ ὥστε διπλασίονα εἶναι τὴν ΑΔ τῆς ΔΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΕΒ, καὶ ἀπὸ τῶν Γ, Δ τῇ ΑΒ πρὸς ὀρθὰς ἤχθωσαν αἱ ΓΕ, ΔΖ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ, ΕΒ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΔ τῆς ΔΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΔ. ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ ΒΑ τῆς ΑΔ. ὡς δὲ ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΖ· ἰσογώνιον γάρ ἐστι τὸ ΑΖΒ τρίγωνον τῷ ΑΖΔ τριγώνῳ· ἡμιόλιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΖ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία

---

[a] For the earlier history of the regular, cosmic or Platonic figures, *v. supra*, pp. 216-225, 378-379.

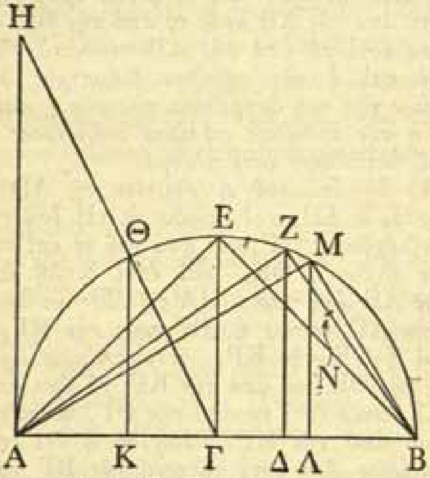
[b] This proposition cannot be fully understood without the previous propositions in the book which it assumes, but it will give an insight into the thoroughness and comprehensiveness of Euclid's methods.

(v.) *Regular Solids* [a]

Euclid, *Elements* xiii. 18 [b]

To set out the sides of the five figures and to compare them one with another.

Let AB, the diameter of the given sphere, be set out, and let it be cut at Γ so that AΓ is equal to ΓB, and at Δ so that AΔ is double of ΔB; and on AB let the semicircle AEB be drawn, and from Γ, Δ let ΓE, ΔZ be drawn at right angles to AB, and let AZ, ZB, EB be joined. Then since $A\Delta = 2\Delta B$, therefore $AB = 3B\Delta$. *Convertendo*, therefore $BA = \frac{3}{2}A\Delta$. But $BA : A\Delta = BA^2 : AZ^2$ [v. Def. 9], for the triangle AZB is equiangular with the triangle AZΔ [vi. 8];



therefore $BA^2 = \frac{3}{2}AZ^2$. But the square on the diameter of the sphere is also one-and-a-half times the

τῆς πλευρᾶς τῆς πυραμίδος. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΑΖ ἄρα ἴση ἐστὶ τῇ πλευρᾷ τῆς πυραμίδος.

Πάλιν, ἐπεὶ διπλασίων ἐστὶν ἡ ΑΔ τῆς ΔΒ, τριπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΔ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΖ· τριπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΖ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίων τῆς τοῦ κύβου πλευρᾶς. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΒΖ ἄρα τοῦ κύβου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, διπλῆ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΕ· διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΕ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίων τῆς τοῦ ὀκταέδρου πλευρᾶς. καί ἐστιν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ ΒΕ ἄρα τοῦ ὀκταέδρου ἐστὶ πλευρά.

Ἤχθω δὴ ἀπὸ τοῦ Α σημείου τῇ ΑΒ εὐθείᾳ πρὸς ὀρθὰς ἡ ΑΗ, καὶ κείσθω ἡ ΑΗ ἴση τῇ ΑΒ, καὶ ἐπεζεύχθω ἡ ΗΓ, καὶ ἀπὸ τοῦ Θ ἐπὶ τὴν ΑΒ κάθετος ἤχθω ἡ ΘΚ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΗΑ τῆς ΑΓ· ἴση γὰρ ἡ ΗΑ τῇ ΑΒ· ὡς δὲ ἡ ΗΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΘΚ πρὸς τὴν ΚΓ, διπλῆ ἄρα καὶ ἡ ΘΚ τῆς ΚΓ. τετραπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΘΚ τοῦ ἀπὸ τῆς ΚΓ· τὰ ἄρα ἀπὸ τῶν ΘΚ, ΚΓ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΘΓ, πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΚΓ. ἴση δὲ ἡ ΘΓ τῇ ΓΒ· πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ. καὶ ἐπεὶ διπλῆ ἐστιν ἡ ΑΒ τῆς ΓΒ, ὧν ἡ ΑΔ τῆς ΔΒ ἐστι διπλῆ, λοιπὴ ἄρα ἡ ΒΔ λοιπῆς

square on the side of the pyramid [xiii. 13]. And AB is the diameter of the sphere; therefore AZ is equal to the side of the pyramid.

Again, since $A\Delta = 2\Delta B$, therefore $AB = 3B\Delta$. But $AB : B\Delta = AB^2 : BZ^2$ [vi. 8, v. Def. 9]; therefore $AB^2 = 3BZ^2$. But the square on the diameter of the sphere is also three times the square on the side of the cube [xiii. 15]. And AB is the diameter of the sphere; therefore BZ is the side of the cube.

And since $A\Gamma = \Gamma B$, therefore $AB = 2B\Gamma$. But $AB : B\Gamma = AB^2 : BE^2$ [vi. 8, v. Def. 9]. Therefore $AB^2 = 2BE^2$. But the square on the diameter of the sphere is also double of the square on the side of the octahedron [xiii. 14]. And AB is the diameter of the given sphere; therefore BE is the side of the octahedron.

Now let AH be drawn from the point A at right angles to the straight line AB, and let AH be made equal to AB, and let HΓ be joined, and from Θ let ΘK be drawn perpendicular to AB. Then since $HA = 2A\Gamma$ (for $HA = AB$), and $HA : A\Gamma = \Theta K : K\Gamma$ [vi. 4], therefore $\Theta K = 2K\Gamma$. Therefore $\Theta K^2 = 4K\Gamma^2$. Therefore $\Theta K^2 + K\Gamma^2 = 5K\Gamma^2 = \Theta\Gamma^2$ [i. 47]. But $\Theta\Gamma = \Gamma B$; therefore $B\Gamma^2 = 5\Gamma K^2$. And since $AB = 2\Gamma B$, and in them $A\Delta = 2\Delta B$, therefore the remainder

τῆς ΔΓ ἐστι διπλῆ. τριπλῆ ἄρα ἡ ΒΓ τῆς ΓΔ· ἐνναπλάσιον ἄρα τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΔ. πενταπλάσιον δὲ τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ· μεῖζον ἄρα τὸ ἀπὸ τῆς ΓΚ τοῦ ἀπὸ τῆς ΓΔ. μείζων ἄρα ἐστὶν ἡ ΓΚ τῆς ΓΔ. κείσθω τῇ ΓΚ ἴση ἡ ΓΛ, καὶ ἀπὸ τοῦ Λ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΛΜ, καὶ ἐπεζεύχθω ἡ ΜΒ. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς ΓΚ, καί ἐστι τῆς μὲν ΒΓ διπλῆ ἡ ΑΒ, τῆς δὲ ΓΚ διπλῆ ἡ ΚΛ, πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΚΛ. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλασίων τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ' οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καί ἐστιν ἡ ΑΒ ἡ τῆς σφαίρας διάμετρος· ἡ ΚΛ ἄρα ἐκ τοῦ κέντρου ἐστὶ τοῦ κύκλου, ἀφ' οὗ τὸ εἰκοσάεδρον ἀναγέγραπται· ἡ ΚΛ ἄρα ἑξαγώνου ἐστὶ πλευρὰ τοῦ εἰρημένου κύκλου. καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος σύγκειται ἔκ τε τῆς τοῦ ἑξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν εἰρημένον κύκλον ἐγγραφομένων, καί ἐστιν ἡ μὲν ΑΒ ἡ τῆς σφαίρας διάμετρος, ἡ δὲ ΚΛ ἑξαγώνου πλευρά, καὶ ἴση ἡ ΑΚ τῇ ΛΒ, ἑκατέρα ἄρα τῶν ΑΚ, ΛΒ δεκαγώνου ἐστὶ πλευρὰ τοῦ ἐγγραφομένου εἰς τὸν κύκλον, ἀφ' οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καὶ ἐπεὶ δεκαγώνου μὲν ἡ ΛΒ, ἑξαγώνου δὲ ἡ ΜΛ· ἴση γάρ ἐστι τῇ ΚΛ, ἐπεὶ καὶ τῇ ΘΚ· ἴσον γὰρ ἀπέχουσιν

[a] Euclid's procedure, in constructing the icosahedron inscribable in a given sphere, is first to construct a circle with radius r such that $r^2 = \frac{1}{5}d^2$, where d is the diameter of the sphere. In this he inscribes a regular decagon, and from its

$B\Delta$ is double of the remainder $\Delta\Gamma$. Therefore $B\Gamma = 3\Gamma\Delta$; therefore $B\Gamma^2 = 9\Gamma\Delta^2$. But $B\Gamma^2 = 5\Gamma K^2$; therefore $\Gamma K^2 > \Gamma\Delta^2$. Therefore $\Gamma K > \Gamma\Delta$. Let $\Gamma\Lambda$ be made equal to ΓK, and from Λ let ΛM be drawn at right angles to AB, and let MB be joined. Then since $B\Gamma^2 = 5\Gamma K^2$, and $AB = 2B\Gamma$, $K\Lambda = 2\Gamma K$, therefore $AB^2 = 5K\Lambda^2$. But the square on the diameter of the sphere is also five times the square on the radius of the circle from which the icosahedron has been described [xiii. 16, coroll.].[a] And AB is the diameter of the sphere; therefore $K\Lambda$ is the radius of the circle from which the icosahedron has been described; therefore $K\Lambda$ is a side of the hexagon in the said circle [iv. 15, coroll.]. And since the diameter of the sphere is made up of the side of the hexagon and two of the sides of the decagon inscribed in the same circle [xiii. 16, coroll.], and AB is the diameter of the sphere, while $K\Lambda$ is the side of the hexagon, and $AK = \Lambda B$, therefore each of the straight lines AK, ΛB is a side of the decagon inscribed in the circle from which the icosahedron has been described. And since ΛB belongs to a decagon and $M\Lambda$ to a hexagon (for $M\Lambda$ is equal to $K\Lambda$ since it is also equal to ΘK,

angular points draws straight lines perpendicular to the plane of the circle and equal in length to r; this determines the angular points of another decagon inscribed in an equal parallel circle. By joining alternate angular points of one decagon, he obtains a pentagon, and then does the same with the other decagon, but in such a manner that the angular points are not opposite one another. Joining the angular points of one pentagon to the nearest angular points of the other, he obtains ten equilateral triangles, which are faces of the icosahedron. He completes the procedure by finding the common vertices of the five equilateral triangles standing on each of the pentagons, which form the remaining faces of the icosahedron.

ἀπὸ τοῦ κέντρου· καί ἐστιν ἑκατέρα τῶν ΘΚ, ΚΛ διπλασίων τῆς ΚΓ· πενταγώνου ἄρα ἐστὶν ἡ ΜΒ. ἡ δὲ τοῦ πενταγώνου ἐστὶν ἡ τοῦ εἰκοσαέδρου· εἰκοσαέδρου ἄρα ἐστὶν ἡ ΜΒ.

Καὶ ἐπεὶ ἡ ΖΒ κύβου ἐστὶ πλευρά, τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Ν, καὶ ἔστω μεῖζον τμῆμα τὸ ΝΒ· ἡ ΝΒ ἄρα δωδεκαέδρου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος ἐδείχθη τῆς μὲν ΑΖ πλευρᾶς τῆς πυραμίδος δυνάμει ἡμιολία, τῆς δὲ τοῦ ὀκταέδρου τῆς ΒΕ δυνάμει διπλασίων, τῆς δὲ τοῦ κύβου τῆς ΖΒ δυνάμει τριπλασίων, οἵων ἄρα ἡ τῆς σφαίρας διάμετρος δυνάμει ἕξ, τοιούτων ἡ μὲν τῆς πυραμίδος τεσσάρων, ἡ δὲ τοῦ ὀκταέδρου τριῶν, ἡ δὲ τοῦ κύβου δύο. ἡ μὲν ἄρα τῆς πυραμίδος πλευρὰ τῆς μὲν τοῦ ὀκταέδρου πλευρᾶς δυνάμει ἐστὶν ἐπίτριτος, τῆς δὲ τοῦ κύβου δυνάμει διπλῆ, ἡ δὲ τοῦ ὀκταέδρου τῆς τοῦ κύβου δυνάμει ἡμιολία. αἱ μὲν οὖν εἰρημέναι τῶν τριῶν σχημάτων πλευραί, λέγω δὴ πυραμίδος καὶ ὀκταέδρου καὶ κύβου, πρὸς ἀλλήλας εἰσὶν ἐν λόγοις ῥητοῖς. αἱ δὲ λοιπαὶ δύο, λέγω δὴ ἥ τε τοῦ εἰκοσαέδρου καὶ ἡ τοῦ δωδεκαέδρου, οὔτε πρὸς ἀλλήλας οὔτε πρὸς τὰς προειρημένας εἰσὶν ἐν

being the same distance from the centre, and each of the straight lines ΘK, KΛ is double of KΓ), therefore MB belongs to a pentagon [xiii. 10, i. 47]. But the side of the pentagon is the side of the icosahedron [xiii. 16]; therefore MB is a side of the icosa hedron.

Now, since ZB is a side of the cube, let it be cut in extreme and mean ratio at N, and let NB be the greater segment; therefore NB is a side of the dodecahedron [xiii. 17, coroll.].[a]

And, since the square on the diameter of the sphere was proved to be one-and-a-half times the square on the side AZ of the pyramid, double of the square on the side BE of the octahedron, and triple of the square on the side ZB of the cube, therefore, of parts of which the square on the diameter of the sphere contains six, the square on the side of the pyramid contains four, the square on the side of the octahedron contains three, and the square on the side of the cube contains two. Therefore the square on the side of the pyramid is four-thirds of the square on the side of the octahedron, and double of the square on the side of the cube; while the square on the side of the octahedron is one-and-a-half times the square on the side of the cube. The said sides of the three figures, I mean the pyramid, the octahedron and the cube, are therefore in rational ratios one to another. But the remaining two, I mean the side of the icosahedron and the side of the dodecahedron, are not in rational ratios either to one another or to the afore-

[a] To construct the dodecahedron inscribable in a given sphere Euclid begins with the cube inscribed in the same sphere, and draws pentagons having the edges of the cube as diagonals.

λόγοις ῥητοῖς· ἄλογοι γάρ εἰσιν, ἡ μὲν ἐλάττων, ἡ δὲ ἀποτομή.

Ὅτι μείζων ἐστὶν ἡ τοῦ εἰκοσαέδρου πλευρὰ ἡ ΜΒ τῆς τοῦ δωδεκαέδρου τῆς ΝΒ, δείξομεν οὕτως.

Ἐπεὶ γὰρ ἰσογώνιόν ἐστι τὸ ΖΔΒ τρίγωνον τῷ ΖΑΒ τριγώνῳ, ἀνάλογόν ἐστιν ὡς ἡ ΔΒ πρὸς τὴν ΒΖ, οὕτως ἡ ΒΖ πρὸς τὴν ΒΑ. καὶ[1] ἐπεὶ τρεῖς εὐθεῖαι ἀνάλογόν εἰσιν, ἔστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας[1]· ἔστιν ἄρα ὡς ἡ ΔΒ πρὸς τὴν ΒΑ, οὕτως τὸ ἀπὸ τῆς ΔΒ πρὸς τὸ ἀπὸ τῆς ΒΖ· ἀνάπαλιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΖΒ πρὸς τὸ ἀπὸ τῆς ΒΔ. τριπλῆ δὲ ἡ ΑΒ τῆς ΒΔ· τριπλάσιον ἄρα τὸ ἀπὸ τῆς ΖΒ τοῦ ἀπὸ τῆς ΒΔ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΑΔ τοῦ ἀπὸ τῆς ΔΒ τετραπλάσιον· διπλῆ γὰρ ἡ ΑΔ τῆς ΔΒ· μεῖζον ἄρα τὸ ἀπὸ τῆς ΑΔ τοῦ ἀπὸ τῆς ΖΒ· μείζων ἄρα ἡ ΑΔ τῆς ΖΒ· πολλῷ ἄρα ἡ ΑΛ τῆς ΖΒ μείζων ἐστίν. καὶ τῆς μὲν ΑΛ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΚΛ, ἐπειδήπερ ἡ μὲν ΛΚ ἑξαγώνου ἐστίν, ἡ δὲ ΚΑ δεκαγώνου· τῆς δὲ ΖΒ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μεῖζον τμῆμά ἐστιν ἡ ΝΒ· μείζων ἄρα ἡ ΚΛ τῆς ΝΒ. ἴση δὲ ἡ ΚΛ τῇ ΛΜ· μείζων ἄρα ἡ ΛΜ τῆς ΝΒ [τῆς δὲ ΛΜ μείζων ἐστὶν ἡ

[1] καὶ ἐπεὶ . . . δευτέρας. "Miramur, cur haec definitio hoc loco omnibus verbis citetur, praesertim forma parum Euclidea, cum tamen antea in hac ipsa propositione toties tacite sit usurpata. itaque puto, verba καὶ ἐπεὶ . . . δευτέρας subditiva esse."—Heiberg.

* If r be the radius of the sphere circumscribing the five regular solids,

said sides; for they are irrational, the one being *minor* [xiii. 16], the other an *apotome* [xiii. 17].[a]

That the side MB of the icosahedron is greater than the side NB of the dodecahedron we shall prove thus.

For since the triangle ZΔB is equiangular with the triangle ZAB [vi. 8], the proportion arises, $\Delta B : BZ = BZ : BA$ [vi. 4]. And since the three straight lines are in proportion, as the first is to the third, so is the square on the first to the square on the second [v. Def. 9]; therefore $\Delta B : BA = \Delta B^2 : BZ^2$; therefore, inversely, $AB : B\Delta = ZB^2 : B\Delta^2$. But $AB = 3B\Delta$; therefore $ZB^2 = 3B\Delta^2$. But $A\Delta^2 = 4\Delta B^2$, for $A\Delta = 2\Delta B$; therefore $A\Delta^2 > ZB^2$; therefore $A\Delta > ZB$; therefore AΛ is by far greater than ZB. And, when AΛ is cut in extreme and mean ratio, KΛ is the greater segment, since ΛK belongs to a hexagon, and KA to a decagon [xiii. 9]; and when ZB is cut in extreme and mean ratio, NB is the greater segment; therefore KΛ is greater than NB. But $K\Lambda = \Lambda M$; therefore $\Lambda M > NB$. Therefore MB,

side of pyramid $= \frac{2}{3}\sqrt{6} \cdot r$

side of octahedron $= \sqrt{2} \cdot r$

side of cube $= \frac{2}{3}\sqrt{3} \cdot r$

side of icosahedron $= \dfrac{r}{5}\sqrt{10(5-\sqrt{5})}$

side of dodecahedron $= \dfrac{r}{3}(\sqrt{15} - \sqrt{3})$.

In the sense of the term irrational as used by Euclid's predecessors and by modern mathematicians, all these expressions are irrational; but in the special sense of Eucl. *Elem.* x. Def. 3, the first three are rational, because their squares are commensurable one with another. The fourth and fifth expressions are irrational even in Euclid's sense, belonging to two species of irrational lines investigated in Book x.

ΜΒ].[1] πολλῷ ἄρα ἡ ΜΒ πλευρὰ οὖσα τοῦ εἰκοσαέδρου μείζων ἐστὶ τῆς ΝΒ πλευρᾶς οὔσης τοῦ δωδεκαέδρου· ὅπερ ἔδει δεῖξαι.

Λέγω δή, ὅτι παρὰ τὰ εἰρημένα πέντε σχήματα οὐ συσταθήσεται ἕτερον σχῆμα περιεχόμενον ὑπὸ ἰσοπλεύρων τε καὶ ἰσογωνίων ἴσων ἀλλήλοις.

Ὑπὸ μὲν γὰρ δύο τριγώνων ἢ ὅλως ἐπιπέδων στερεὰ γωνία οὐ συνίσταται. ὑπὸ δὲ τριῶν τριγώνων ἡ τῆς πυραμίδος, ὑπὸ δὲ τεσσάρων ἡ τοῦ ὀκταέδρου, ὑπὸ δὲ πέντε ἡ τοῦ εἰκοσαέδρου· ὑπὸ δὲ ἓξ τριγώνων ἰσοπλεύρων τε καὶ ἰσογωνίων πρὸς ἑνὶ σημείῳ συνισταμένων οὐκ ἔσται στερεὰ γωνία· οὔσης γὰρ τῆς τοῦ ἰσοπλεύρου τριγώνου γωνίας διμοίρου ὀρθῆς ἔσονται αἱ ἓξ τέσσαρσιν ὀρθαῖς ἴσαι· ὅπερ ἀδύνατον· ἅπασα γὰρ στερεὰ γωνία ὑπὸ ἐλασσόνων ἢ τεσσάρων ὀρθῶν περιέχεται. διὰ τὰ αὐτὰ δὴ οὐδὲ ὑπὸ πλειόνων ἢ ἓξ γωνιῶν ἐπιπέδων στερεὰ γωνία συνίσταται.

Ὑπὸ δὲ τετραγώνων τριῶν ἡ τοῦ κύβου γωνία περιέχεται· ὑπὸ δὲ τεσσάρων ἀδύνατον· ἔσονται γὰρ πάλιν τέσσαρες ὀρθαί.

Ὑπὸ δὲ πενταγώνων ἰσοπλεύρων καὶ ἰσογωνίων, ὑπὸ μὲν τριῶν ἡ τοῦ δωδεκαέδρου· ὑπὸ δὲ τεσσάρων ἀδύνατον· οὔσης γὰρ τῆς τοῦ πενταγώνου ἰσοπλεύρου γωνίας ὀρθῆς καὶ πέμπτου, ἔσονται αἱ τέσσαρες γωνίαι τεσσάρων ὀρθῶν μείζους· ὅπερ ἀδύνατον.

Οὐδὲ μὴν ὑπὸ πολυγώνων ἑτέρων σχημάτων περισχεθήσεται στερεὰ γωνία διὰ τὸ αὐτὸ ἄτοπον.

Οὐκ ἄρα παρὰ τὰ εἰρημένα πέντε σχήματα

[1] τῆς . . . ΜΒ del. Heiberg.

which is a side of the icosahedron, is much greater than NB, which is a side of the dodecahedron; which was to be proved.

I say now that *no other figure, besides the said five figures, can be constructed so as to be contained by equilateral and equiangular figures equal one to another.*

For a solid angle cannot be constructed out of two triangles, or, generally, planes. With three triangles there is constructed the angle of the pyramid, with four the angle of the octahedron, with five the angle of the icosahedron; but no solid angle can be formed by placing together at one point six equilateral and equiangular triangles; for inasmuch as the angle of the equilateral triangle is two-thirds of a right angle, the six will be equal to four right angles; which is impossible, for any solid angle is contained by angles less than four right angles [xi. 21]. For the same reasons no solid angle can be constructed out of more than six plane angles.

By three squares the angle of the cube is contained; but it is impossible for a solid angle to be contained by four squares; for they will again be four right angles [xi. 21].

By three equilateral and equiangular pentagons the angle of the dodecahedron is contained; but by four it is impossible for a solid angle to be contained; for inasmuch as the angle of the equilateral pentagon is a right angle and a fifth, the four angles will be greater than four right angles; which is impossible [xi. 21].

Nor will a solid angle be contained by any other polygonal figures by reason of the same absurdity.

Therefore no other figure, besides the said five

ἕτερον σχῆμα στερεὸν συνταθήσεται ὑπὸ ἰσοπλεύρων τε καὶ ἰσογωνίων περιεχόμενον· ὅπερ ἔδει δεῖξαι.

(c) The Data

Eucl., ed. Heiberg-Menge vi. 2. 1-15

Ὅροι

αʹ. Δεδομένα τῷ μεγέθει λέγεται χωρία τε καὶ γραμμαὶ καὶ γωνίαι, οἷς δυνάμεθα ἴσα πορίσασθαι.

βʹ. Λόγος δεδόσθαι λέγεται, ᾧ δυνάμεθα τὸν αὐτὸν πορίσασθαι.

γʹ. Εὐθύγραμμα σχήματα τῷ εἴδει δεδόσθαι λέγεται, ὧν αἵ τε γωνίαι δεδομέναι εἰσὶ κατὰ μίαν καὶ οἱ λόγοι τῶν πλευρῶν πρὸς ἀλλήλας δεδομένοι.

δʹ. Τῇ θέσει δεδόσθαι λέγονται σημεῖά τε καὶ γραμμαὶ καὶ γωνίαι, ἃ τὸν αὐτὸν ἀεὶ τόπον ἐπέχει.

εʹ. Κύκλος τῷ μεγέθει δεδόσθαι λέγεται, οὗ δέδοται ἡ ἐκ τοῦ κέντρου τῷ μεγέθει.

ϛʹ. Τῇ θέσει δὲ καὶ τῷ μεγέθει κύκλος δεδόσθαι λέγεται, οὗ δέδοται τὸ μὲν κέντρον τῇ θέσει, ἡ δὲ ἐκ τοῦ κέντρου τῷ μεγέθει.

(d) The Porisms

Procl. *in Eucl.* i., ed Friedlein 301. 21-302. 13 ; Eucl., ed. Heiberg-Menge viii. 237. 9-27

Ἕν τι τῶν γεωμετρικῶν ἐστιν ὀνομάτων τὸ πόρισμα. τοῦτο δὲ σημαίνει διττόν· καλοῦσι γὰρ

---

[a] Euclid's *Data* (Δεδομένα) is his only work in pure geometry to have survived in Greek apart from the *Elements*. (His book *On Divisions of Figures* has survived in Arabic, *v. supra*, p. 156 n. *c*.) It is closely connected with Books i.-vi. of the *Elements*, and its general character will be suffi-

figures, can be constructed so as to be contained by equilateral and equiangular figures; which was to be proved.

(c) The Data[a]

Eucl., ed. Heiberg-Menge vi. 2. 1-15

Definitions

1. Areas, lines and angles are said to be given in magnitude when we can make others equal to them.
2. A ratio is said to be given when we can make another equal to it.
3. Rectilineal figures are said to be given in species when their angles are severally given and the ratios of the sides one towards another are also given.
4. Points, lines and angles are said to be given in position when they always occupy the same place.
5. A circle is said to be given in magnitude when the radius is given in magnitude.
6. A circle is said to be given in position and in magnitude when the centre is given in position and the radius in magnitude.

(d) The Porisms

Proclus, *On Euclid* i., ed. Friedlein 301. 21–302. 13; Eucl., ed. Heiberg-Menge viii. 237. 9-27

Porism is one of the terms used in geometry. It has a twofold meaning. For porisms are in the first

ciently indicated by these first few definitions. The object of a proposition called a *datum* is to prove that, if in a figure certain properties are given, other properties are also given, in one or other of the senses defined in the definitions. Pappus included the book in his Τόπος ἀναλυόμενος (*Treasury of Analysis*).

πορίσματα, καὶ ὅσα θεωρήματα συγκατασκευάζεται ταῖς ἄλλων ἀποδείξεσιν οἷον ἕρμαια καὶ κέρδη τῶν ζητούντων ὑπάρχοντα, καὶ ὅσα ζητεῖται μέν, εὑρέσεως δὲ χρῄζει καὶ οὔτε γενέσεως μόνης οὔτε θεωρίας ἁπλῆς. ὅτι μὲν γὰρ τῶν ἰσοσκελῶν αἱ πρὸς τῇ βάσει ἴσαι θεωρῆσαι δεῖ, καὶ ὄντων δή τινων[1] πραγμάτων ἐστὶν ἡ τοιαύτη γνῶσις. τὴν δὲ γωνίαν δίχα τεμεῖν ἢ τρίγωνον συστήσασθαι ἢ ἀφελεῖν ἢ προσθέσθαι,[2] ταῦτα πάντα ποίησίν τινος ἀπαιτεῖ· τοῦ δὲ δοθέντος κύκλου τὸ κέντρον εὑρεῖν, ἢ δύο δοθέντων συμμέτρων μεγεθῶν τὸ μέγιστον καὶ κοινὸν μέτρον εὑρεῖν, ἢ ὅσα τοιάδε, μεταξύ πώς ἐστι προβλημάτων καὶ θεωρημάτων. οὔτε γὰρ γενέσεις εἰσὶν ἐν τούτοις τῶν ζητουμένων, ἀλλ' εὑρέσεις, οὔτε θεωρία ψιλή. δεῖ γὰρ ὑπ' ὄψιν ἀγαγεῖν καὶ πρὸ ὀμμάτων ποιήσασθαι τὸ ζητούμενον. τοιαῦτα ἄρα ἐστὶν καὶ ὅσα Εὐκλείδης πορίσματα γέγραφε, γ̅ βιβλία Πορισμάτων συντάξας.

Papp. *Coll.* vii., ed. Hultsch 648. 18–660. 16; Eucl., ed. Heiberg-Menge viii. 238. 10–243. 5

Μετὰ δὲ τὰς Ἐπαφὰς ἐν τρισὶ βιβλίοις Πορίσματά ἐστιν Εὐκλείδου, πολλοῖς ἄθροισμα φιλοτεχνότατον εἰς τὴν ἀνάλυσιν τῶν ἐμβριθεστέρων προβλημάτων . . .

[1] τινων Heiberg, τῶν codd.
[2] προσθέσθαι Heiberg, θέσθαι codd.

---

[a] A porism in this sense is commonly called a *corollary*.

[b] Euclid's *Porisms* has unfortunately not survived, which is a great misfortune as it appears to have been the most original and advanced of all his works. Our knowledge of its contents comes solely from Pappus.

[c] Pappus is describing the books comprised in his Τόπος ἀναλυόμενος (*Treasury of Analysis*). He proceeds to give an

place such theorems as can be established by means of the proofs of other theorems, being a kind of windfall or bonus in the investigation[a]; and in the second place porisms are things which are sought, but need some finding, being neither brought into existence simply nor yet investigated by theory alone. For to prove that the angles at the base of an isosceles triangle are equal is a matter for theoretic inquiry only, and such knowledge is of certain things already in existence. But to bisect an angle or to construct a triangle, to cut off or to add—all these things require the making of something; and to find the centre of a given circle, or to find the greatest common measure of two given commensurable magnitudes, and so on, is in some way intermediate between problems and theorems. For in these cases there is no bringing into existence of the things sought, but a finding of them; nor is the inquiry pure theory. For it is necessary to bring what is sought into view and to exhibit it before the eyes. To this class belong the porisms which Euclid wrote and arranged in his three books of *Porisms*.[b]

Pappus, *Collection* vii., ed. Hultsch 648. 18–660. 16 : Eucl. ed. Heiberg-Menge viii. 238. 10–243. 5

After the *Contacts* (of Apollonius) come, in three books, the *Porisms* of Euclid, a collection most skilfully framed, in the opinion of many, for the analysis of the more weighty problems[c] . . .

explanation of the term *porism* as used by Euclid with which Proclus's account is in substantial agreement. In addition, he gave another definition by "more recent geometers" (*ὑπὸ τῶν νεωτέρων*), viz., "a porism is that which falls short of a locus-theorem in respect of its hypothesis" (*πόρισμά ἐστιν τὸ λεῖπον ὑποθέσει τοπικοῦ θεωρήματος*).

Περιλαβεῖν δὲ πολλὰ μιᾷ προτάσει ἥκιστα δυνατὸν ἐν τούτοις διὰ τὸ καὶ αὐτὸν Εὐκλείδην οὐ πολλὰ ἐξ ἑκάστου εἴδους τεθεικέναι, ἀλλὰ δείγματος ἕνεκα ἐκ τῆς πολυπληθείας ἓν ἢ[1] ὀλίγα. πρὸς ἀρχῇ δὲ ὅμως[2] τοῦ πρώτου βιβλίου τέθεικεν ὁμοειδῆ τινα[3] ἐκείνου τοῦ δαψιλεστέρου εἴδους τῶν τόπων, ὡς ῑ τὸ πλῆθος. διὸ καὶ περιλαβεῖν ταύτας μιᾷ προτάσει ἐνδεχόμενον εὑρόντες οὕτως ἐγράψαμεν· ἐὰν ὑπτίου ἢ παρυπτίου τρία τὰ ἐπὶ μιᾶς σημεῖα [ἢ παραλλήλου τῆς ἑτέρας τὰ δύο][4] δεδομένα ᾖ, τὰ δὲ λοιπὰ πλὴν ἑνὸς ἅπτηται θέσει δεδομένης εὐθείας, καὶ τοῦθ' ἅψεται θέσει δεδομένης εὐθείας. τοῦτ' ἐπὶ τεσσάρων μὲν εὐθειῶν εἴρηται μόνων, ὧν οὐ πλείονες ἢ δύο διὰ τοῦ αὐτοῦ σημείου εἰσίν, ἀγνοεῖται δὲ ἐπὶ παντὸς τοῦ προ-

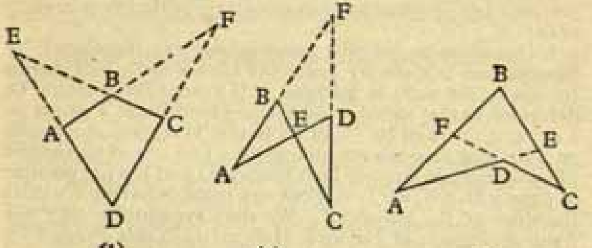
[1] ἓν ἢ Littré, ἕνα Hultsch.
[2] δὲ ὅμως Heiberg, δεδομένον cod. (sequente lacuna) del. Hultsch.
[3] τινα Heiberg, πῶν cod., πάντ' Hultsch.
[4] ἢ . . . δύο interpolatori trib. Hultsch.

---

[a] The four straight lines are described in the Greek as (the sides) ὑπτίου ἢ παρυπτίου, *i.e.*, as the sides of *supine* and *hyper-supine* quadrilaterals. Robert Simson (*Opera quaedam reliqua*, p. 348) explains a ὕπτιον σχῆμα as being of the

Now to comprehend many propositions in one enunciation is far from easy in these porisms, because Euclid himself has not given many of each species, but out of a great number he has selected one or a few by way of example. But at the beginning of the first book he has given certain allied propositions, ten in number, from that more abundant species consisting of loci. Finding that these can be comprehended in one enunciation, we have therefore written it out in this manner : *If, in a system of four straight lines which cut one another two and two, the three points [of intersection] on one straight line be given, while the rest except one lie on different straight lines given in position, the remaining point also will be on a straight line given in position.*[a] This has been enunciated in the case of four straight lines only, of which not more than two pass through the same point, and it is not

nature of (1) in the accompanying diagrams, while (2) and (3) are παρύπτια σχήματα. He also explained the correct



(1) (2) (3)

meaning of the rather loose proviso, τὰ δὲ λοιπὰ πλὴν ἑνὸς ἅπτηται θέσει δεδομένης εὐθείας. Applied to these figures, the enunciation states that if A, B, F are given, while the loci of C and D are straight lines, then the locus of E is also a straight line.

τεινομένου πλήθους ἀληθὲς ὑπάρχον οὕτως λεγόμενον· ἐὰν ὁποσαιοῦν εὐθεῖαι τέμνωσιν ἀλλήλας, μὴ πλείονες ἢ δύο διὰ τοῦ αὐτοῦ σημείου, πάντα δὲ ἐπὶ μιᾶς αὐτῶν δεδομένα ᾖ, καὶ τῶν ἐπὶ ἑτέρας ἕκαστον ἅπτηται θέσει δεδομένης εὐθείας, ἢ καθολικωτέραν οὕτως· ἐὰν ὁποσαιοῦν εὐθεῖαι τέμνωσιν ἀλλήλας, μὴ πλείονες ἢ δύο διὰ τοῦ αὐτοῦ σημείου, πάντα δὲ τὰ ἐπὶ μιᾶς αὐτῶν σημεῖα δεδομένα ᾖ, τῶν δὲ λοιπῶν τὸ πλῆθος ἐχόντων τρίγωνον ἀριθμὸν ἡ πλευρὰ τούτου ἕκαστον ἔχῃ σημεῖον ἁπτόμενον εὐθείας θέσει δεδομένης, τῶν τριῶν μὴ πρὸς γωνίαις ὑπαρχόντων τριγώνου χωρίου, ἕκαστον λοιπὸν σημεῖον ἅψεται θέσει δεδομένης εὐθείας. τὸν δὲ Στοιχειωτὴν οὐκ εἰκὸς ἀγνοῆσαι τοῦτο, τὴν δ' ἀρχὴν μόνην τάξαι. . . .

Ἔχει δὲ τὰ τρία βιβλία τῶν Πορισμάτων λήμματα λη, αὐτὰ δὲ θεωρημάτων ἐστὶν ροα.

[a] *Sc.* a triangle having as its sides three of the given straight lines.

[b] The meaning of this enunciation was discovered by Simson, and is given by Loria (*Le scienze esatte nell' antica Grecia*, p. 256 n. 3) as follows: "If a complete n-lateral be deformed so that its sides respectively turn about n points on a straight line, and $(n-1)$ of its $\frac{1}{2}n(n-1)$ vertices move each on a straight line, the remaining $\frac{1}{2}(n-1)(n-2)$ of its vertices likewise move on straight lines; provided that it is not possible to form with the $(n-1)$ vertices any triangle having for sides the sides of the polygon." We may sympathize with the frank confession of Edmond Halley (*Apollonii Pergaei De sectione rationis*, p. xxxvii) that he could make no sense out of this passage.

generally known that it is true of any assigned number of straight lines when thus enunciated : *If any number of straight lines cut one another, not more than two passing through the same point, and all the points [of intersection] on one of them be given, and if each of those which are on another lie on a straight line given in position*—or still more generally in this manner : *If any number of straight lines cut one another, not more than two passing through the same point, and all the points [of intersection] on one of them be given, while of the remaining points of intersection, in multitude equal to a triangular number, a number corresponding to the side of this triangular number lie respectively on straight lines given in position, provided that of these latter points no three are at the vertices of a triangle,*[a] *each of the remaining points will lie on a straight line given in position.*[b] The writer of the *Elements* was probably not unaware of this, but he merely laid down the principle.[c] . . .

The three books of the *Porisms* involve 38 lemmas[d] ; of the theorems themselves there are 171.[e]

[c] Pappus proceeds to state in order 28 propositions from Euclid's work.

[d] Pappus gives these lemmas to the *Porisms* (Pappus, ed. Hultsch 866. 1–918. 20 ; Eucl. ed. Heiberg-Menge viii. 243. 10–274. 10).

[e] The reconstruction of the *Porisms* has been one of the most fascinating inquiries pursued by students of Greek mathematics, and thereby Chasles was led to the idea of anharmonic ratios. Further details will be found in Loria, *loc. cit.*, pp. 253-265, Heath, *H.G.M.* i. 431-438, and I am greatly indebted to the translations and notes in these works.

(e) The Conics

Papp. *Coll.* vii. 30-36, ed. Hultsch 672. 18-678. 24

Τὰ Εὐκλείδου βιβλία δ̄ Κωνικῶν Ἀπολλώνιος ἀναπληρώσας καὶ προσθεὶς ἕτερα δ̄ παρέδωκεν η̄ Κωνικῶν τεύχη. Ἀρισταῖος δέ, ὃς γέγραφε τὰ μέχρι τοῦ νῦν ἀναδιδόμενα στερεῶν τόπων τεύχη ε̄ συνεχῆ τοῖς κωνικοῖς, ἐκάλει [καὶ οἱ πρὸ Ἀπολλωνίου][1] τῶν τριῶν κωνικῶν γραμμῶν τὴν μὲν ὀξυγωνίου, τὴν δὲ ὀρθογωνίου, τὴν δὲ ἀμβλυγωνίου κώνου τομήν. . . . ὃν δέ φησιν [sc. Ἀπολλώνιος] ἐν τῷ τρίτῳ τόπον ἐπὶ γ̄ καὶ δ̄ γραμμὰς μὴ τελειωθῆναι ὑπὸ Εὐκλείδου, οὐδ' ἂν αὐτὸς ἠδυνήθη οὐδ' ἄλλος οὐδεὶς ἀλλ' οὐδὲ μικρόν τι προσθεῖναι τοῖς ὑπὸ Εὐκλείδου γραφεῖσιν[2] διά γε μόνων τῶν προδεδειγμένων ἤδη κωνικῶν ἄχρι τῶν κατ' Εὐκλείδην, ὡς καὶ αὐτὸς μαρτυρεῖ λέγων ἀδύνατον εἶναι τελειωθῆναι, χωρὶς ὧν αὐτὸς προγράφειν ἠναγκάσθη. ὁ δὲ Εὐκλείδης ἀποδεχόμενος τὸν Ἀρισταῖον ἄξιον ὄντα ἐφ' οἷς ἤδη παραδεδώκει κωνικοῖς, καὶ μὴ φθάσας ἢ μὴ θελήσας ἐπικαταβάλλεσθαι τούτων τὴν αὐτὴν πραγματείαν, ἐπιεικέστατος ὢν καὶ πρὸς ἅπαντας εὐμενὴς τοὺς καὶ κατὰ ποσὸν συναύξειν δυναμένους τὰ μαθήματα, ὡς δεῖ, καὶ μηδαμῶς προσκρουστικὸς ὑπάρχων, καὶ ἀκριβὴς μὲν οὐκ ἀλαζονικὸς δὲ καθάπερ οὗτος, ὅσον δυνατὸν ἦν δεῖξαι τοῦ τόπου διὰ τῶν ἐκείνου

[1] καὶ οἱ πρὸ Ἀπολλωνίου del. Hultsch.
[2] ἀλλ' . . . γραφεῖσιν del. Hultsch.

[a] Euclid's *Conics* has not survived, but an idea of its contents can be obtained from Archimedes' references to propositions proved in the *Elements of Conics* (ἐν τοῖς κωνικοῖς

(c) The Conics [a]

Pappus, *Collection* vii. 30-36, ed. Hultsch 672. 18–678. 24

Apollonius, who completed the four books of Euclid's *Conics* and added another four, gave us eight books of *Conics*. Aristaeus, who wrote the still extant [b] five books of *Solid Loci* supplementary to the *Conics*, called the three conics sections of an acute-angled, right-angled and obtuse-angled cone respectively. . . . Apollonius says in his third book that the "locus with respect to three or four lines" had not been fully worked out by Euclid, and in fact neither Apollonius himself nor anyone else could have added anything to what Euclid wrote, using only those properties of conics which had been proved up to Euclid's time; as Apollonius himself bears witness when he says that the locus could not be fully investigated without the propositions that he had been compelled to work out for himself. Now Euclid regarded Aristaeus as deserving credit for his contributions to conics, and did not try to anticipate him or to overthrow his system; for he showed scrupulous fairness and exemplary kindness towards all who were able in any degree to advance mathematics, and was never offensive, but aimed at accuracy, and did not boast like the other. Accordingly he wrote so much about the locus as was possible by means of

στοιχείοις), a term which would cover the treatises both of Aristaeus and of Euclid. The *Surface-Loci* and the *Porisms* of Euclid appear to have contained further developments in the theory of conics.

[b] This has been taken to imply that Euclid's *Conics* was already lost when Pappus wrote. Nothing more is known of this Aristaeus, unless he is identical with the Aristaeus said by Hypsicles (Eucl. ed. Heiberg-Menge v. 6. 22-23) to have written a book called *Comparison of the Five Regular Solids*.

Κωνικῶν ἔγραψεν, οὐκ εἰπὼν τέλος ἔχειν τὸ δεικνύμενον. τότε γὰρ ἦν ἀναγκαῖον ἐξελέγκειν, νῦν δ᾿ οὐδαμῶς, ἐπείτοι καὶ αὐτὸς ἐν τοῖς Κωνικοῖς ἀτελῆ τὰ πλεῖστα καταλιπὼν οὐκ εὐθύνεται. προσθεῖναι δὲ τῷ τόπῳ τὰ λειπόμενα δεδύνηται προφαντασιωθεὶς τοῖς ὑπὸ Εὐκλείδου γεγραμμένοις ἤδη περὶ τοῦ τόπου καὶ συσχολάσας τοῖς ὑπὸ Εὐκλείδου μαθηταῖς ἐν Ἀλεξανδρείᾳ πλεῖστον χρόνον, ὅθεν ἔσχε καὶ τὴν τοιαύτην ἕξιν οὐκ ἀμαθῆ.

Οὗτος δὲ ὁ ἐπὶ $\bar{\gamma}$ καὶ $\bar{\delta}$ γραμμὰς τόπος, ἐφ᾿ ᾧ μέγα φρονεῖ προσθεὶς χάριν ὀφείλειν εἰδέναι τῷ πρώτῳ γράψαντι, τοιοῦτός ἐστιν.[1] ἐὰν γάρ, θέσει δεδομένων τριῶν εὐθειῶν, ἀπό τινος τοῦ αὐτοῦ[2] σημείου καταχθῶσιν ἐπὶ τὰς τρεῖς ἐν δεδομέναις γωνίαις εὐθεῖαι, καὶ λόγος ᾖ δοθεὶς τοῦ ὑπὸ δύο κατηγμένων περιεχομένου ὀρθογωνίου πρὸς τὸ ἀπὸ τῆς λοιπῆς τετράγωνον, τὸ σημεῖον ἅψεται θέσει δεδομένου στερεοῦ τόπου, τουτέστιν μιᾶς τῶν τριῶν κωνικῶν γραμμῶν. καὶ ἐὰν ἐπὶ $\bar{\delta}$ εὐθείας θέσει δεδομένας καταχθῶσιν εὐθεῖαι ἐν δεδομέναις γωνίαις, καὶ λόγος ᾖ δοθεὶς τοῦ ὑπὸ δύο κατηγμένων πρὸς τὸ ὑπὸ τῶν λοιπῶν δύο κατηγμένων, ὁμοίως τὸ σημεῖον ἅψεται θέσει δεδομένης κώνου τομῆς.

[1] ὁ δὲ Εὐκλείδης . . . τοιοῦτός ἐστιν "scholiastae cuidam historiae quidem veterum mathematicorum non imperito, sed qui dicendi genere languido et inconcinno usus sit" tribuit Hultsch.

[2] τοῦ αὐτοῦ del. Hultsch.

[a] The three-line locus is, of course, a particular example of the four-line locus. It seems clear that Apollonius himself did not have a complete solution of the four-line locus, but

the *Conics* of Aristaeus, but did not claim finality for his proofs. If he had done so, we should have been obliged to censure him, but as things are he is in no wise to blame, seeing that Apollonius himself is not called to account, though he left the most part of his *Conics* incomplete. Moreover Apollonius was able to add the lacking portion of the theory of the locus through having become familiar beforehand with what had been written about it by Euclid, and through having spent much time with Euclid's pupils at Alexandria, whence he derived his scientific habit of mind.

Now this "locus with respect to three and four lines," the theory of which he is so proud of having expanded—though he ought rather to acknowledge his debt to the original author—is of this kind. If three straight lines be given in position, and from one and the same point straight lines be drawn to meet the three straight lines at given angles, and if the ratio of the rectangle contained by two of the straight lines towards the square on the remaining straight line be given, then the point will lie on a solid locus given in position, that is on one of the three conic sections. And if straight lines be drawn to meet at given angles four straight lines given in position, and the ratio of the rectangle contained by two of the straight lines so drawn towards the rectangle contained by the remaining two be given, then in the same way the point will lie on a conic section given in position.[a]

his *Conics* iii. 53-56 [Props. 74-76] amounts to a demonstration of the converse of the three-line locus, viz., *if from any point of a conic there be drawn three straight lines in fixed directions to meet respectively two fixed tangents to the conic and their chord of contact, the ratio of the rectangle contained*

Eucl. *Phaen.* Praef., Eucl. ed. Heiberg-Menge viii. 6. 5-7

Ἐὰν γὰρ κῶνος ἢ κύλινδρος ἐπιπέδῳ τμηθῇ μὴ παρὰ τὴν βάσιν, ἡ τομὴ γίγνεται ὀξυγωνίου κώνου τομή, ἥτις ἐστὶν ὁμοία θυρεῷ.

(*f*) The Surface-Loci

Papp. *Coll.* vii., ed. Hultsch 636. 23-24

Εὐκλείδου Τόπων τῶν πρὸς ἐπιφανείᾳ β̄.

Procl. *in Eucl.* i., ed. Friedlein 394. 16–395. 2

Καλῶ δὲ τοπικὰ μέν, ὅσοις ταὐτὸν σύμπτωμα πρὸς ὅλῳ τινὶ τόπῳ συμβέβηκεν, τόπον δὲ γραμμῆς ἢ ἐπιφανείας θέσιν ποιοῦσαν ἓν καὶ ταὐτὸν σύμπτωμα. τῶν γὰρ τοπικῶν τὰ μέν ἐστι πρὸς γραμμαῖς συνιστάμενα, τὰ δὲ πρὸς ἐπιφανείαις. καὶ ἐπειδὴ τῶν γραμμῶν αἱ μέν εἰσιν ἐπίπεδοι, αἱ δὲ στερεαί—ἐπίπεδοι μέν, ὧν ἐν ἐπιπέδῳ ἁπλῆ ἡ νόησις, ὡς τῆς εὐθείας, στερεαὶ δέ, ὧν ἡ γένεσις ἔκ τινος τομῆς ἀναφαίνεται στερεοῦ σχήματος, ὡς τῆς κυλινδρικῆς ἕλικος καὶ τῶν κωνικῶν γραμμῶν

---

by the first two lines so drawn to the square on the third line is constant. For a solution and full discussion of the four-line locus, reference should be made to Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, pp. 126 ff., or Heath, *Apollonius of Perga*, pp. cxxxviii-cl.

[a] Euclid's *Phenomena* is an astronomical work largely based on two treatises by Autolycus of Pitane (c. 315–240 B.C.) which are also extant.

[b] Menaechmus is believed to have discovered the conic sections as sections of a right-angled, acute-angled and obtuse-angled cone respectively by a plane perpendicular

Euclid, Preface to *Phenomena*,[a] Eucl. ed. Heiberg-Menge viii. 6. 5-7

If a cone or cylinder be cut by a plane not parallel to the base, the resulting section is a section of an acute-angled cone which is similar to a shield.[b]

(*f*) The Surface-Loci

Pappus, *Collection* vii., ed. Hultsch 636. 23-24

Euclid's two books of *Surface-Loci*.[c]

Proclus, *On Euclid* i., ed. Friedlein 394. 16–395. 2

I call locus-theorems those which deal with the same property throughout the whole of a locus, and a locus I call a position of a line or surface which has throughout one and the same property. Some locus-theorems are constructed on lines and others on surfaces. Furthermore, since lines may be plane or solid—plane being those which are simply generated in a plane, like the straight line, and solid those which are generated from some section of a solid figure, like the cylindrical helix or the conic sections

to a generating line. This passage shows that Euclid, at least, was also aware that an ellipse could be obtained as a section of a right cylinder by a plane not parallel to the base, and the fact may well have been known before his time; Heiberg (*Literärgeschichtliche Studien über Euklid*, p. 88) thinks that Menaechmus probably used θυρεός as the name for the ellipse.

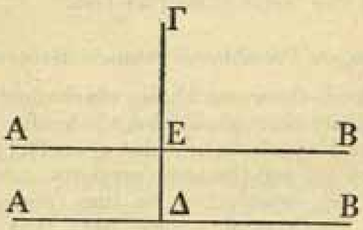
[c] This entry is taken from the list of books in Pappus's Τόπος ἀναλυόμενος (*Treasury of Analysis*). The work is lost, but we can conjecture what surface-loci were from remarks by Proclus and Pappus himself, and we can get some idea of the contents of Euclid's treatise from two lemmas given to it by Pappus.

—φαίην ἂν καὶ τῶν πρὸς γραμμαῖς τοπικῶν τὰ μὲν ἐπίπεδον ἔχειν τόπον, τὰ δὲ στερεόν.

Papp. *Coll.* vii. 312-316, ed. Hultsch 1004. 16–1010. 15 ;
Eucl. ed. Heiberg-Menge viii. 274. 18-278. 15

Εἰς τοὺς πρὸς ἐπιφανείᾳ

α΄. Ἐὰν ᾖ εὐθεῖα ἡ ΑΒ καὶ παρὰ θέσει ἡ ΓΔ, καὶ ᾖ λόγος τοῦ ὑπὸ ΑΔΒ πρὸς τὸ ἀπὸ ΔΓ, τὸ Γ



ἅπτεται κωνικῆς γραμμῆς. ἐὰν οὖν ἡ μὲν ΑΒ στερηθῇ τῆς θέσεως, καὶ τὰ Α, Β στερηθῇ τοῦ δοθέντα[1] εἶναι, γένηται δὲ πρὸς θέσει εὐθείαις[2] ταῖς ΑΕ, ΕΒ, τὸ Γ μετεωρισθὲν γίνεται πρὸς θέσει ἐπιφανείᾳ. τοῦτο δὲ ἐδείχθη.

β΄. Ἐὰν ᾖ θέσει εὐθεῖα ἡ ΑΒ καὶ δοθὲν τὸ Γ

[1] δοθέντα Heiberg, δοθέντος cod., Hultsch.
[2] εὐθείαις Tannery, εὐθεῖα cod.

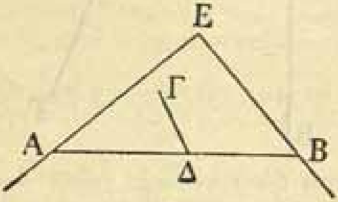
[a] From this passage, confirmed by Eutocius, line-loci would appear to be loci which *are* lines, and surface-loci would seem to be loci which *are* surfaces. Pappus, in *Coll.* iv. 33, ed. Hultsch 258. 20-25, implies, however, that surface-loci are loci *traced* on surfaces, and he gives the cylindrical helix as an example of such a locus. *Cf. supra*, p. 348 n. *a*.

—it would appear that line-loci may be plane loci or solid loci.[a]

Pappus, *Collection* vii. 312-316, ed. Hultsch 1004. 16–1010. 15; Eucl. ed. Heiberg-Menge viii. 274. 18–278. 15

Lemmas to the Surface-Loci

1. If AB be a straight line and ΓΔ be parallel to a straight line given in position, and if the ratio



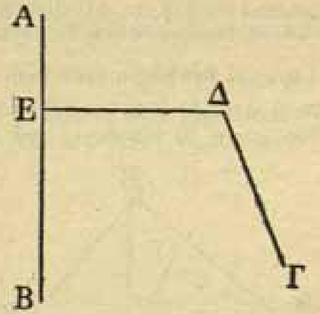
AΔ . ΔB : ΔΓ² be given, the point Γ lies on a conic section. If AB be no longer given in position and A, B be no longer given but lie on straight lines AE, EB given in position, the point Γ raised above [the plane containing AE, EB] is on a surface given in position. And this was proved.[b]

2. If AB be a straight line given in position, and

[b] The Greek text and the figure in it (given on the left-hand page) are unsatisfactory, but Tannery pointed out that by reading εὐθείαις instead of εὐθεῖα a satisfactory meaning can be obtained (*Bulletin des sciences mathématiques*, 2e série, vi. 149-150). He also indicated the correct figure, which was first printed by Zeuthen (*Die Lehre von den Kegelschnitten im Altertum*, pp. 423-430). *The Works of Archimedes*, by T. L. Heath, pp. lxii-lxiv, should also be consulted.

The first sentence states one of the fundamental properties of conic sections. A literal translation of the opening words in the second sentence would run: "If AB be deprived of its position, and the points A, B be deprived of their character

ἐν τῷ αὐτῷ ἐπιπέδῳ, καὶ διαχθῇ ἡ ΔΓ, καὶ πρὸς ὀρθὰς[1] ἀχθῇ ἡ ΔΕ, λόγος δὲ ᾖ τῆς ΓΔ πρὸς ΔΕ,



τὸ Δ ἅπτεται θέσει κωνικῆς τομῆς· δεικτέον[2] δέ, ὅτι γραμμῆς ⟨μέρος ποιεῖ τὸν τόπον⟩.[3] δειχθήσεται δὲ οὕτως προγραφέντος τόπου[4] τοῦδε.

γʹ. Δύο δοθέντων τῶν Α, Β καὶ ὀρθῆς τῆς ΓΔ λόγος ἔστω τοῦ ἀπὸ ΑΔ πρὸς τὰ ἀπὸ ΓΔ, ΔΒ. λέγω, ὅτι τὸ Γ ἅπτεται κώνου τομῆς, ἐάν τε ᾖ ὁ λόγος ἴσος πρὸς ἴσον ἢ μείζων πρὸς ἐλάσσονα ἢ ἐλάσσων πρὸς μείζονα.

Ἔστω γὰρ πρότερον ὁ λόγος ἴσος πρὸς ἴσον. καὶ ἐπεὶ ἴσον ἐστὶν τὸ ἀπὸ ΑΔ τοῖς ἀπὸ ΓΔ, ΔΒ, κείσθω τῇ ΒΔ ἴση ἡ ΔΕ. ἴσον ἄρα ἐστὶ τὸ ὑπὸ ΒΑΕ τῷ ἀπὸ ΔΓ. τετμήσθω δίχα ἡ ΑΒ τῷ Ζ·

[1] πρὸς ὀρθὰς Hultsch, παρὰ θέσει cod.
[2] δεικτέον Hultsch in adn., δείκνυται cod.
[3] μέρος ποιεῖ τὸν τόπον add. Gerhardt, Hultsch.
[4] τόπου "immo τοῦ λήμματος" Hultsch.

of being given . . ." The text leaves it uncertain whether, when AB is no longer given in position, it remains constant

the point Γ be given in the same plane, and ΔΓ be drawn, and ΔE be drawn perpendicular [to the given straight line AB], and if the ratio ΓΔ : ΔE be given, the point Δ will lie on a conic section.[a] But it must be shown that part of the curve forms the locus. This will be proved as follows by means of this lemma.

3. Given[b] the two points A, B and the perpendicular ΓΔ, let the ratio $A\Delta^2 : \Gamma\Delta^2 + \Delta B^2$ be given. I say that the point Γ lies on a conic section, whether the ratio be of equal to equal, or greater to less, or less to greater.

For in the first place let the ratio be of equal to equal. Since $A\Delta^2 = \Gamma\Delta^2 + \Delta B^2$, let ΔE be made equal to BΔ.

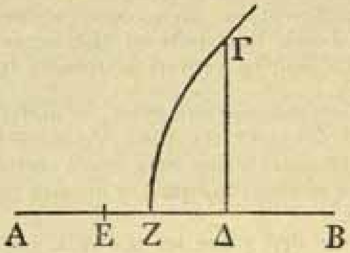
Then [$BA \cdot AE + E\Delta^2 = A\Delta^2$ [Eucl. ii. 6
$= \Gamma\Delta^2 + \Delta B^2$ [*ex. hyp.*,
and so] $BA \cdot AE = \Gamma\Delta^2$.

in length or varies. Zeuthen conjectures that two cases were considered by Euclid: (1) AB remains of constant length, while AE, EB are parallel instead of meeting in a point; and (2) AE, EB meet in a point and AB always moves parallel to itself, so varying in length. In the former case Γ lies on the surface described by a conic section moving bodily, in the latter case the surface is a cone.

[a] This is the definition of a conic in terms of its focus and directrix, AB being the directrix, Γ the focus, Δ any point on the curve, and the ratio ΓΔ : ΔE the eccentricity of the conic. Since Pappus proves this property for all three conics by transforming it to the more familiar axial form, it must have been assumed by Euclid without proof, and was presumably first demonstrated by Aristaeus. This is all the more remarkable as the focus-directrix property is nowhere mentioned by Apollonius, and, indeed, is found in only two other places in the whole of the Greek mathematical writings, v. *supra*, p. 362 n. *a*.

[b] Diagram on p. 496.

δοθὲν ἄρα τὸ Ζ. καὶ ἔσται διπλῆ ἡ ΑΕ τῆς ΖΔ ὥστε τὸ ὑπὸ ΒΑΕ τὸ δίς ἐστιν ὑπὸ τῶν ΑΒ, ΖΔ.



καί ἐστιν ἡ διπλῆ τῆς ΑΒ δοθεῖσα· τὸ ἄρα ὑπὸ δοθείσης καὶ τῆς ΖΔ ἴσον ἐστὶν τῷ ἀπὸ τῆς ΔΓ. τὸ Γ ἄρα ἅπτεται θέσει παραβολῆς ἐρχομένης διὰ τοῦ Ζ.

δʹ. Συντεθήσεται δὴ ὁ τόπος οὕτως·

Ἔστω τὰ δοθέντα Α, Β, ὁ δὲ λόγος ἔστω ἴσος πρὸς ἴσον, καὶ τετμήσθω ἡ ΑΒ δίχα τῷ Ζ, τῆς δὲ ΑΒ διπλῆ ἔστω ἡ Ρ, καὶ θέσει οὔσης εὐθείας τῆς ΖΒ πεπερασμένης κατὰ τὸ Ζ, τῆς δὲ Ρ δεδομένης τῷ μεγέθει, γεγράφθω περὶ ἄξονα τὸν ΖΒ παραβολὴ ἡ ΗΖ, ὥστε, οἷον ἐὰν ἐπ' αὐτῆς σημεῖον ληφθῇ ὡς τὸ Γ, κάθετος δὲ ἀχθῇ ἡ ΓΔ, ἴσον εἶναι τὸ ὑπὸ Ρ, ΖΔ, τῷ ἀπὸ ΔΓ. καὶ ἤχθω ὀρθὴ ἡ ΒΗ. λέγω, ὅτι τὸ ΓΗ μέρος τῆς παραβολῆς ἐστιν.[1]

Ἤχθω γὰρ κάθετος ἡ ΓΔ, καὶ τῇ ΒΔ ἴση κείσθω ἡ ΔΕ. ἐπεὶ οὖν διπλῆ ἐστιν ἡ μὲν ΑΒ τῆς ΒΖ, ἡ δὲ ΕΒ τῆς ΒΔ, διπλῆ ἄρα καὶ ἡ ΑΕ τῆς ΖΔ· τὸ ἄρα ὑπὸ ΒΑΕ ἴσον ἐστὶν τῷ δὶς ὑπὸ τῶν ΑΒ,

Let AB be bisected at Z; the point Z is therefore given.

And AE $[= AB - EB$
$= 2BZ - 2B\Delta]$
$= 2Z\Delta$.

Therefore $BA \, . \, AE = 2BA \, . \, Z\Delta$,

[and so $2BA \, . \, Z\Delta = \Gamma\Delta^2$].

Now 2BA is given; therefore the rectangle contained by a given straight line and $Z\Delta$ is equal to the square on $\Delta\Gamma$. Therefore the point Γ lies on a parabola passing through Z.

4. The synthesis of the locus is accomplished in this way.[a]

Let the given points be A, B, let the ratio be of equal to equal, let AB be bisected at Z, let P be double of AB; and since ZB with an end point Z is a straight line given in position, and P is given in magnitude, with ZB as axis, let there be drawn [Apoll. *Conics* i. 52] the parabola HZ, such that, if any point Γ be taken upon it, and the perpendicular $\Gamma\Delta$ be drawn, the rectangle contained by P, $Z\Delta$ is equal to the square on $\Delta\Gamma$; and let the perpendicular BH be drawn. I say that ΓH is a part of the parabola [forming the locus].

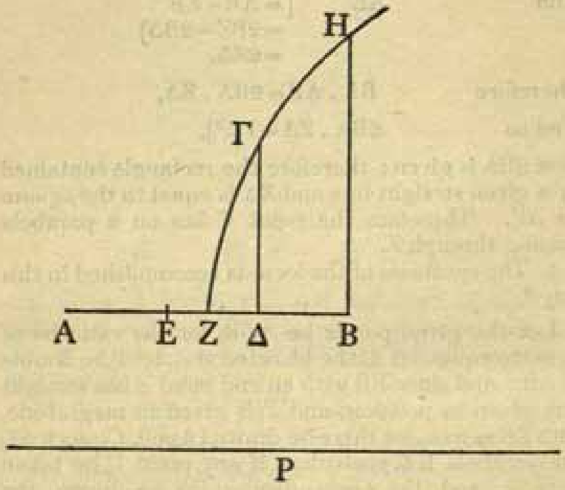
For let the perpendicular $\Gamma\Delta$ be drawn, and let ΔE be made equal to $B\Delta$. Then since $AB = 2BZ$, $EB = 2B\Delta$, therefore $AE \, [= AB - EB] = 2Z\Delta$;

therefore $BA \, . \, AE = 2AB \, . \, Z\Delta$
$= \Delta\Gamma^2$. [by construction

[a] Diagram on p. 498.

[1] ἐστιν; we should expect ποιεῖ τὸν τόπον.

ΖΔ, τουτέστιν τῷ ἀπὸ ΔΓ. κοινὸν προσκείσθω τὸ ἀπὸ ΕΔ ἴσον ὂν τῷ ἀπὸ ΔΒ· ὅλον ἄρα τὸ ἀπὸ



ΑΔ ἴσον ἐστὶν τοῖς ἀπὸ τῶν ΓΔ, ΔΒ. ἡ ΖΓΗ ἄρα γραμμὴ ποιεῖ τὸν τόπον.

ε΄. Ἔστω δὴ πάλιν τὰ δύο δοθέντα σημεῖα τὰ Α, Β, καὶ εὐθεῖά τε ἡ ΔΓ καὶ ὀρθή,[1] λόγος δὲ ἔστω τοῦ ἀπὸ ΑΔ πρὸς τὰ ἀπὸ ΒΔ, ΔΓ ἐπὶ μὲν τῆς πρώτης πτώσεως μείζων πρὸς ἐλάσσονα,[2] ἐπὶ δὲ τῆς δευτέρας ἐλάσσων πρὸς μείζονα[3]· λέγω, ὅτι τὸ Γ ἅπτεται κώνου τομῆς, ἐπὶ μὲν τῆς πρώτης πτώσεως ἐλλείψεως, ἐπὶ δὲ τῆς δευτέρας ὑπερβολῆς.

Ἐπεὶ γὰρ λόγος ἐστὶν τοῦ ἀπὸ ΑΔ πρὸς τὰ ἀπὸ ΒΔ, ΔΓ, ὁ αὐτὸς αὐτῷ γεγονέτω ὁ τοῦ ἀπὸ ΕΔ

Let the equals $E\Delta^2$, ΔB^2 be added to either side;

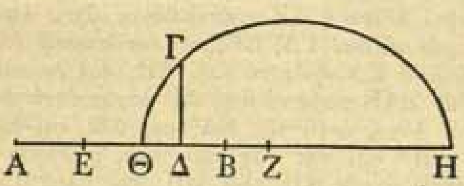
then $\qquad [BA \cdot AE + E\Delta^2 = \Gamma\Delta^2 + \Delta B^2$

and so] $\qquad A\Delta^2 = \Gamma\Delta^2 + \Delta B^2$. $\qquad$ [Eucl. ii. 6

Therefore the curve $Z\Gamma H$ forms the locus.

5. Again, let the two given points be A, B, and let $\Delta\Gamma$ be a perpendicular straight line, and let the ratio $A\Delta^2 : B\Delta^2 + \Delta\Gamma^2$ be in the first case the ratio of a greater to a less, and in the second case of a less to a greater. I say, that the point Γ lies on a conic section, which is in the first case an ellipse and in the second case a hyperbola.[a]

Since the given ratio is $A\Delta^2 : B\Delta^2 + \Gamma\Delta^2$, let [E be taken on AB so that] $E\Delta^2 : \Delta B^2$ be in the same



[a] The Greek text from this point onwards is unsatisfactory, and contains mathematical errors which Commandinus and Hultsch corrected. The demonstration also leaves many gaps which I have filled, again following those commentators.

---

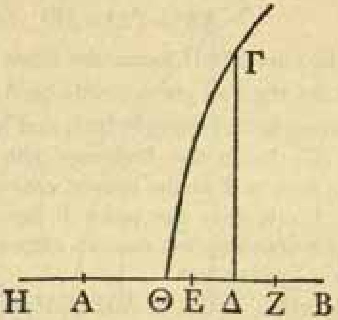
[1] *εὐθεῖά τε ἡ* ΔΓ *καὶ ὀρθή* Heiberg: *κατήχθω ὀρθὴ ἡ* ΔΓ, Commandinus, Hultsch: *ἐφάπτεται ἡ* ΔΓ *καὶ ὀρθή* cod.

[2] *μείζων πρὸς ἐλάσσονα* Hultsch, *ἐλάσσων πρὸς μείζονα* cod.

[3] *ἐλάσσων πρὸς μείζονα* Hultsch, *μείζων πρὸς ἐλάσσονα* cod.

[4] EΔ Hultsch, BΔ cod.

πρὸς τὸ ἀπὸ ΔΒ.[1] ἐπὶ μὲν οὖν τῆς πρώτης πτώσεως ἐλάσσων ἐστὶν ἡ ΒΔ τῆς ΔΕ, ἐπὶ δὲ τῆς



δευτέρας μείζων ἐστὶν ἡ ΒΔ τῆς ΔΕ. κείσθω οὖν τῇ ΕΔ ἴση ἡ ΔΖ. ἐπεὶ λόγος ἐστὶν τοῦ ἀπὸ ΑΔ πρὸς τὰ ἀπὸ ΓΔ, ΔΒ, καί ἐστιν αὐτῷ ὁ αὐτὸς ὁ τοῦ ἀπὸ ΕΔ πρὸς τὸ ἀπὸ ΔΒ, καὶ λοιπὸς ἄρα τοῦ ὑπὸ ΖΑΕ πρὸς τὸ ἀπὸ ΔΓ λόγος ἐστὶν δοθείς. ἐπεὶ δὲ λόγος ἐστὶν τῆς ΕΔ πρὸς ΔΒ [καὶ τῆς ΖΔ πρὸς ΔΒ][2] καὶ τῆς ΖΒ πρὸς ΒΔ, ὁ αὐτὸς αὐτῷ γεγονέτω ὁ τῆς ΑΒ πρὸς ΒΗ· καὶ ὅλης ἄρα τῆς ΑΖ πρὸς ΔΗ λόγος ἐστὶν δοθείς. πάλιν, ἐπεὶ λόγος ἐστὶν τῆς ΕΔ πρὸς ΔΒ δοθείς, [καὶ τῆς ΕΒ ἄρα πρὸς ΒΔ λόγος ἐστὶν δοθείς].[3] ὁ αὐτὸς αὐτῷ γεγονέτω ὁ τῆς ΑΘ[4] πρὸς ΒΘ· λόγος ἄρα καὶ τῆς ΑΒ πρὸς ΒΘ ἐστιν δοθείς· [δοθὲν ἄρα τὸ Θ].[5] καὶ λοιπὸς τῆς ΑΕ πρὸς ΘΔ λόγος ἐστὶν δοθείς· καὶ τοῦ ὑπὸ ΖΑΕ ἄρα πρὸς τὸ ὑπὸ ΘΔΗ λόγος ἐστὶ δοθείς. τοῦ δὲ ὑπὸ ΖΑΕ πρὸς τὸ ἀπὸ ΓΔ λόγος ἐστὶν δοθείς· καὶ τοῦ ὑπὸ ΗΔΘ ἄρα πρὸς τὸ ἀπὸ

[1] ΔΒ Hultsch, ΔΕ cod.   [2] καὶ . . . πρὸς ΔΒ del. Hultsch.

ratio; then in the first case $B\Delta$ is less than ΔE, while in the second case $B\Delta$ is greater than ΔE. Let ΔZ be made equal to $E\Delta$. Since the given ratio is $A\Delta^2 : \Gamma\Delta^2 + \Delta B^2$, and $E\Delta^2 : \Delta B^2$ is equal to it, the ratio

$$[A\Delta^2 - E\Delta^2 : \Gamma\Delta^2 + \Delta B^2 - \Delta B^2,$$

that is, by Eucl. ii. 6,]

$$ZA \,.\, AE : \Delta\Gamma^2$$

is given. Now since the ratio $[E\Delta^2 : \Delta B^2$ is given, therefore] $E\Delta : \Delta B$ is given, therefore $[\Delta Z : \Delta B$ is given. Accordingly, in the first case $\Delta Z : BZ$, and therefore $BZ : \Delta B$, is given; in the second case, because $\Delta Z : \Delta B$ or inversely $\Delta B : \Delta Z$ is given, therefore $\Delta B : BZ$ or inversely] $BZ : \Delta B$ is given. Let [H be taken on AB produced so that] $AB : BH = BZ : \Delta B$. Then [in the first case $AB + BZ : BH + \Delta B$, in the second case $AB - BZ : BH - \Delta B$, that is in either case] $AZ : \Delta H$ is given. Let [Θ be taken on AB such that] $A\Theta : B\Theta = E\Delta : \Delta B$. Then the ratio $AB : B\Theta$ is given. And [because by construction $A\Theta : B\Theta = E\Delta : B\Delta$, *componendo* $A\Theta + B\Theta : B\Theta = E\Delta + B\Delta : B\Delta$, or $AB : B\Theta = EB : \Delta B$. Therefore $AB - EB : B\Theta - \Delta B$, that is,] $AE : \Theta\Delta$ is given. [Now $AZ : \Delta H$ was given;] therefore $AE \,.\, AZ : \Theta\Delta \,.\, \Delta H$ is given. But $ZA \,.\, AE : \Delta\Gamma^2$ was given; therefore the ratio $H\Delta \,.\, \Delta\Theta : \Delta\Gamma^2$ is given. [But the point Δ is given, and by construction the points E, Z are given; and because $AB : BH = BZ : \Delta B$ and also

---

[3] καὶ . . . δοθείς del. Hultsch.
[4] AΘ Hultsch, AB cod.
[5] δοθὲν ἄρα τὸ Θ del. Hultsch.

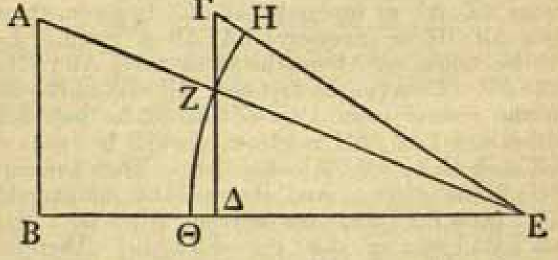
ΔΓ λόγος ἐστὶν δοθείς. καί ἐστιν δύο δοθέντα τὰ Θ, Η· ἐπὶ μὲν ἄρα τῆς πρώτης πτώσεως τὸ Γ ἅπτεται ἐλλείψεως, ἐπὶ δὲ τῆς δευτέρας ὑπερβολῆς.

(g) The Optics

Eucl. *Optic.* 8, Eucl. ed. Heiberg-Menge vii. 14. 1–16. 5

Τὰ ἴση μεγέθη καὶ παράλληλα ἄνισον διεστηκότα ἀπὸ τοῦ ὄμματος οὐκ ἀναλόγως τοῖς διαστήμασιν ὁρᾶται.

Ἔστω δύο μεγέθη τὰ ΑΒ, ΓΔ ἄνισον διεστηκότα ἀπὸ τοῦ ὄμματος τοῦ Ε. λέγω, ὅτι οὐκ ἔστιν, ὡς



φαίνεται ἔχον, ὡς τὸ ΓΔ πρὸς τὸ ΑΒ, οὕτως τὸ ΒΕ πρὸς τὸ ΕΔ. προσπιπτέτωσαν γὰρ ἀκτῖνες αἱ ΑΕ, ΕΓ, καὶ κέντρῳ μὲν τῷ Ε διαστήματι δὲ τῷ ΕΖ κύκλου γεγράφθω περιφέρεια ἡ ΗΖΘ. ἐπεὶ οὖν τὸ ΕΖΓ τρίγωνον τοῦ ΕΖΗ τομέως μεῖζόν ἐστιν, τὸ δὲ ΕΖΔ τρίγωνον τοῦ ΕΖΘ τομέως ἔλαττόν ἐστιν, τὸ ΕΖΓ ἄρα τρίγωνον πρὸς τὸν

[a] Pappus proceeds to make the formal synthesis, as in the case of the parabola, and then formally proves his original

$A\Theta : B\Theta = E\Delta : \Delta B$, therefore] the points H, Θ are also given. [Therefore in the first case HΘ is the diameter of an ellipse, in the second it is the diameter of a hyperbola; and] therefore the point Γ lies in the first case on an ellipse, in the second on a hyperbola.[a]

(g) The Optics[b]

Euclid, *Optics* 8, Eucl. ed. Heiberg-Menge vii. 14. 1–16. 5

The apparent sizes of equal and parallel magnitudes at unequal distances from the eye are not proportional to those distances.

Let AB, ΓΔ be the two magnitudes at unequal distances from the eye, E. I say that the ratio of the apparent size of ΓΔ to the apparent size of AB is not equal to the ratio of BE to EΔ. For let the rays AE, EΓ fall,[c] and with centre E and radius EZ let the arc of a circle, HZΘ, be drawn. Then since the triangle EZΓ is greater than the sector EZH, while the triangle EZΔ is less than the sector EZΘ, therefore

proposition in the case where the locus is a parabola; the proof where the locus is an ellipse or hyperbola has been lost, but can easily be supplied.

[b] Euclid's *Optics* exists in two recensions, both contained in vol. vii. of the Heiberg-Menge edition of Euclid's works. One is the recension of Theon, but Heiberg discovered in Viennese and Florentine mss. an earlier and markedly different recension, and there is every reason to believe it is Euclid's own work; it is from this earlier text that the proposition here quoted is given. The *Optics* is an elementary treatise on perspective. It is based on some false physical hypotheses, but has some interesting mathematical theorems.

[c] Euclid, like Plato, believed [*Optics*, Def. 1] that rays of light proceed from the eye to the object, and not from the object to the eye.

ΕΖΗ τομέα μείζονα λόγον ἔχει ἤπερ τὸ ΕΖΔ τρίγωνον πρὸς τὸν ΕΖΘ τομέα. καὶ ἐναλλὰξ τὸ ΕΖΓ τρίγωνον πρὸς τὸ ΕΖΔ τρίγωνον μείζονα λόγον ἔχει ἤπερ ὁ ΕΖΗ τομεὺς πρὸς τὸν ΕΖΘ τομέα, καὶ συνθέντι τὸ ΕΓΔ τρίγωνον πρὸς τὸ ΕΖΔ τρίγωνον μείζονα λόγον ἔχει ἤπερ ὁ ΕΗΘ τομεὺς πρὸς τὸν ΕΖΘ τομέα. ἀλλ' ὡς τὸ ΕΔΓ πρὸς τὸ ΕΖΔ τρίγωνον, οὕτως ἡ ΓΔ πρὸς τὴν ΔΖ. ἡ δὲ ΓΔ τῇ ΑΒ ἐστιν ἴση, καὶ ὡς ἡ ΑΒ πρὸς τὴν ΔΖ, ἡ ΒΕ πρὸς τὴν ΕΔ.[a] ἡ ΒΕ ἄρα πρὸς τὴν ΕΔ μείζονα λόγον ἔχει ἤπερ ὁ ΕΗΘ τομεὺς πρὸς τὸν ΕΖΘ τομέα. ὡς δὲ ὁ τομεὺς πρὸς τὸν τομέα, οὕτως ἡ ὑπὸ ΗΕΘ γωνία πρὸς τὴν ὑπὸ ΖΕΘ γωνίαν. ἡ ΒΕ ἄρα πρὸς τὴν ΕΔ μείζονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΗΕΘ γωνία πρὸς τὴν ὑπὸ ΖΕΘ. καὶ ἐκ μὲν τῆς ὑπὸ ΗΕΘ γωνίας βλέπεται τὸ ΓΔ, ἐκ δὲ τῆς ὑπὸ ΖΕΘ τὸ ΑΒ. οὐκ ἀνάλογον ἄρα τοῖς ἀποστήμασιν ὁρᾶται τὰ ἴσα μεγέθη.

---

[a] This is equivalent, of course, to saying that

$$\frac{\tan \mathrm{HE\Theta}}{\tan \mathrm{ZE\Theta}} > \frac{\text{angle } \mathrm{ZE\Theta}}{\text{angle } \mathrm{HE\Theta}},$$

a well-known theorem in trigonometry; the full expression

triangle EZΓ : sector EZH > triangle EZΔ : sector EZΘ.

Invertendo,

triangle EZΓ : triangle EZΔ > sector EZH : sector EZΘ,

and *componendo,*

triangle EΓΔ : triangle EZΔ > sector EHΘ : sector EZΘ.

But triangle EΓΔ : triangle EZΔ = ΓΔ : ΔZ.

Now ΓΔ = AB, and AB : ΔZ = BE : EΔ.

Therefore BE : EΔ > sector EHΘ : sector EZΘ.

Now

sector EHΘ : sector EZΘ = angle HEΘ : angle ZEΘ.

Therefore

BE : EΔ > angle HEΘ : angle ZEΘ.[a]

And ΓΔ is seen in the angle HEΘ, while AB is seen in the angle ZEΘ. Therefore[b] the apparent sizes of equal magnitudes are not proportional to their distances.

of the theorem is: If α, β are two angles such that $\alpha < \beta < \frac{1}{2}\pi$, then

$$\frac{\tan\alpha}{\tan\beta} < \frac{\alpha}{\beta}.$$

[b] By Def. 4, which asserts: "Things seen under a greater angle appear greater, and those seen under a lesser angle appear less, while things seen under equal angles appear equal."

Printed in Great Britain by R. & R. Clark, Limited, *Edinburgh*

THE LOEB CLASSICAL LIBRARY

VOLUMES ALREADY PUBLISHED

LATIN AUTHORS

Ammianus Marcellinus. J. C. Rolfe. 3 Vols. (Vols. I and II *2nd Imp. revised.*)
Apuleius: The Golden Ass (Metamorphoses). W. Adlington (1566). Revised by S. Gaselee. (*7th Imp.*)
St. Augustine, Confessions of. W. Watt (1631). 2 Vols. (Vol. I *7th Imp.*, Vol. II *6th Imp.*)
St. Augustine, Select Letters. J. H. Baxter.
Ausonius. H. G. Evelyn White. 2 Vols. (Vol. II *2nd Imp.*)
Bede. J. E. King. 2 Vols.
Boethius: Tracts and De Consolatione Philosophiae. Rev. H. F. Stewart and E. K. Rand. (*4th Imp.*)
Caesar: Civil Wars. A. G. Peskett. (*4th Imp.*)
Caesar: Gallic War. H. J. Edwards. (*9th Imp.*)
Cato and Varro: De Re Rustica. H. B. Ash and W. D. Hooper. (*2nd Imp.*)
Catullus. F. W. Cornish; Tibullus. J. B. Postgate; and Pervigilium Veneris. J. W. Mackail. (*12th Imp.*)
Celsus: De Medicina. W. G. Spencer. 3 Vols. (Vol. I *3rd Imp. revised.*)
Cicero: Brutus and Orator. G. L. Hendrickson and H. M. Hubbell. (*2nd Imp.*)
Cicero: De Fato; Paradoxa Stoicorum; De Partitione Oratoria. H. Rackham. (With De Oratore, Vol. II.) (*2nd Imp.*)

Cicero: De Finibus. H. Rackham. *(3rd Imp. revised.)*
Cicero: De Inventione, etc. H. M. Hubbell.
Cicero: De Natura Deorum and Academica. H. Rackham.
Cicero: De Officiis. Walter Miller. *(4th Imp.)*
Cicero: De Oratore. E. W. Sutton and H. Rackham. 2 Vols. *(2nd Imp.)*
Cicero: De Republica and De Legibus. Clinton W. Keyes. *(3rd Imp.)*
Cicero: De Senectute, De Amicitia, De Divinatione. W. A. Falconer. *(5th Imp.)*
Cicero: In Catilinam, Pro Murena, Pro Sulla, Pro Flacco. Louis E. Lord. *(2nd Imp. revised.)*
Cicero: Letters to Atticus. E. O. Winstedt. 3 Vols. (Vol. I *6th Imp.*, Vols. II and III *3rd Imp.*)
Cicero: Letters to his Friends. W. Glynn Williams. 3 Vols. (Vols. I and II *2nd Imp.*)
Cicero: Philippics. W. C. A. Ker. *(2nd Imp.)*
Cicero: Pro Archia, Post Reditum, De Domo, De Haruspicum Responsis, Pro Plancio. N. H. Watts. *(2nd Imp.)*
Cicero: Pro Caecina, Pro Lege Manilia, Pro Cluentio, Pro Rabirio. H. Grose Hodge. *(2nd Imp.)*
Cicero: Pro Milone, In Pisonem, Pro Scauro, Pro Fonteio, Pro Rabirio Postumo, Pro Marcello, Pro Ligario, Pro Rege Deiotaro. N. H. Watts.
Cicero: Pro Quinctio, Pro Roscio Amerino, Pro Roscio Comoedo, Contra Rullum. J. H. Freese. *(2nd Imp.)*
Cicero: Tusculan Disputations. J. E. King. *(3rd Imp.)*
Cicero: Verrine Orations. L. H. G. Greenwood. 2 Vols. (Vol. I *2nd Imp.*)
Claudian. M. Platnauer. 2 Vols.
Columella: De Re Rustica. H. B. Ash. 2 Vols. Vol. I. Books I-IV. *(2nd Imp.)*
Curtius, Q.: History of Alexander. J. C. Rolfe. 2 Vols.
Florus. E. S. Forster; and Cornelius Nepos. J. C. Rolfe. *(2nd Imp.)*
Frontinus: Stratagems and Aqueducts. C. E. Bennett and M. B. McElwain. *(2nd Imp.)*
Fronto: Correspondence. C. R. Haines. 2 Vols.
Gellius. J. C. Rolfe. 3 Vols. (Vols. I and II *2nd Imp.*)
Horace: Odes and Epodes. C. E. Bennett. *(13th Imp. revised.)*

HORACE: SATIRES, EPISTLES, ARS POETICA. H. R. Fairclough. (*8th Imp. revised.*)
JEROME: SELECT LETTERS. F. A. Wright.
JUVENAL AND PERSIUS. G. G. Ramsay. (*7th Imp.*)
LIVY. B. O. Foster, F. G. Moore, Evan T. Sage and A. C. Schlesinger. 13 Vols. Vols. I-XII. (Vol. I *3rd Imp.*, Vols. II-V, VII, IX-XII *2nd Imp. revised.*)
LUCAN. J. D. Duff. (*2nd Imp.*)
LUCRETIUS. W. H. D. Rouse. (*6th Imp. revised.*)
MARTIAL. W. C. A. Ker. 2 Vols. (Vol. I *5th Imp.*, Vol. II *4th Imp. revised.*)
MINOR LATIN POETS: from PUBLILIUS SYRUS to RUTILIUS NAMATIANUS, including GRATTIUS, CALPURNIUS SICULUS, NEMESIANUS, AVIANUS, with "Aetna," "Phoenix" and other poems. J. Wight Duff and Arnold M. Duff. (*2nd Imp.*)
OVID: THE ART OF LOVE AND OTHER POEMS. J. H. Mozley. (*3rd Imp.*)
OVID: FASTI. Sir James G. Frazer. (*2nd Imp.*)
OVID: HEROIDES AND AMORES. Grant Showerman. (*4th Imp.*)
OVID: METAMORPHOSES. F. J. Miller. 2 Vols. (Vol. I *9th Imp.*, Vol. II *7th Imp.*)
OVID: TRISTIA AND EX PONTO. A. L. Wheeler. (*2nd Imp.*)
PETRONIUS. M. Heseltine; SENECA: APOCOLOCYNTOSIS. W. H. D. Rouse. (*7th Imp. revised.*)
PLAUTUS. Paul Nixon. 5 Vols. (Vol. I *5th Imp.*, Vols. II and III *4th Imp.*)
PLINY: LETTERS. Melmoth's translation revised by W. M. L. Hutchinson. 2 Vols. (Vol. I *5th Imp.*, Vol. II *4th Imp.*)
PLINY: NATURAL HISTORY. H. Rackham and W. H. S. Jones. 10 Vols. Vols. I-V. (Vols. I-III *2nd Imp.*)
PROPERTIUS. H. E. Butler. (*5th Imp.*)
PRUDENTIUS. H. J. Thomson. 2 Vols. Vol. I.
QUINTILIAN. H. E. Butler. 4 Vols. (*2nd Imp.*)
REMAINS OF OLD LATIN. E. H. Warmington. 4 Vols. Vol. I (Ennius and Caecilius). Vol. II (Livius, Naevius, Pacuvius, Accius). Vol. III (Lucilius, Laws of the XII Tables). Vol. IV (Archaic Inscriptions). (Vol. IV *2nd Imp.*)
SALLUST. J. C. Rolfe. (*3rd Imp. revised.*)

SCRIPTORES HISTORIAE AUGUSTAE. D. Magie. 3 Vols. (Vol. I *2nd Imp. revised.*)
SENECA: APOCOLOCYNTOSIS. *Cf.* PETRONIUS.
SENECA: EPISTULAE MORALES. R. M. Gummere. 3 Vols. (Vol. I *3rd Imp.*, Vols. II and III *2nd Imp. revised.*)
SENECA: MORAL ESSAYS. J. W. Basore. 3 Vols. (Vol. II *3rd Imp. revised*, Vol. III *2nd Imp. revised.*)
SENECA: TRAGEDIES. F. J. Miller. 2 Vols. (Vol. I *3rd Imp.*, Vol. II *2nd Imp. revised.*)
SIDONIUS: POEMS AND LETTERS. W. B. Anderson. 2 Vols. Vol. I.
SILIUS ITALICUS. J. D. Duff. 2 Vols. (Vol. I *2nd Imp.*, Vol. II *3rd Imp.*)
STATIUS. J. H. Mozley. 2 Vols.
SUETONIUS. J. C. Rolfe. 2 Vols. (Vol. I *7th Imp.*, Vol. II *6th Imp.*)
TACITUS: DIALOGUS. Sir Wm. Peterson; and AGRICOLA AND GERMANIA. Maurice Hutton. (*6th Imp.*)
TACITUS: HISTORIES AND ANNALS. C. H. Moore and J. Jackson. 4 Vols. (Vols. I and II *2nd Imp.*)
TERENCE. John Sargeaunt. 2 Vols. (Vol. I *6th Imp.*, Vol. II *5th Imp.*)
TERTULLIAN: APOLOGIA AND DE SPECTACULIS. T. R. Glover; MINUCIUS FELIX. G. H. Rendall.
VALERIUS FLACCUS. J. H. Mozley. (*2nd Imp. revised.*)
VARRO: DE LINGUA LATINA. R. G. Kent. 2 Vols. (*2nd Imp. revised.*)
VELLEIUS PATERCULUS AND RES GESTAE DIVI AUGUSTI. F. W. Shipley.
VIRGIL. H. R. Fairclough. 2 Vols. (Vol. I *17th Imp.*, Vol. II *13th Imp. revised.*)
VITRUVIUS: DE ARCHITECTURA. F. Granger. 2 Vols. (Vol. I *2nd Imp.*)

GREEK AUTHORS

ACHILLES TATIUS. S. Gaselee. (*2nd Imp.*)
AENEAS TACTICUS, ASCLEPIODOTUS AND ONASANDER. The Illinois Greek Club. (*2nd Imp.*)
AESCHINES. C. D. Adams. (*2nd Imp.*)

AESCHYLUS. H. Weir Smyth. 2 Vols. (Vol. I *5th Imp.*, Vol. II *4th Imp.*)

ALCIPHRON, AELIAN AND PHILOSTRATUS: LETTERS. A. R. Benner and F. H. Fobes.

APOLLODORUS. Sir James G. Frazer. 2 Vols. (*2nd Imp.*)

APOLLONIUS RHODIUS. R. C. Seaton. (*4th Imp.*)

THE APOSTOLIC FATHERS. Kirsopp Lake. 2 Vols. (*7th Imp.*)

APPIAN'S ROMAN HISTORY. Horace White. 4 Vols. (Vol. I *3rd Imp.*, Vols. II, III and IV *2nd Imp.*)

ARATUS. *Cf.* CALLIMACHUS.

ARISTOPHANES. Benjamin Bickley Rogers. 3 Vols. (Vols. I and II *5th Imp.*, Vol. III *4th Imp.*) Verse trans.

ARISTOTLE: ART OF RHETORIC. J. H. Freese. (*3rd Imp.*)

ARISTOTLE: ATHENIAN CONSTITUTION, EUDEMIAN ETHICS, VIRTUES AND VICES. H. Rackham. (*2nd Imp.*)

ARISTOTLE: GENERATION OF ANIMALS. A. L. Peck. (*2nd Imp.*)

ARISTOTLE: METAPHYSICS. H. Tredennick. 2 Vols. (*3rd Imp.*)

ARISTOTLE: MINOR WORKS. W. S. Hett. "On Colours," "On Things Heard," "Physiognomics," "On Plants," "On Marvellous Things Heard," "Mechanical Problems," "On Indivisible Lines," "Situations and Names of Winds," "On Melissus, Xenophanes, and Gorgias."

ARISTOTLE: NICOMACHEAN ETHICS. H. Rackham. (*5th Imp. revised.*)

ARISTOTLE: OECONOMICA AND MAGNA MORALIA. G. C. Armstrong. (With Metaphysics, Vol. II.) (*3rd Imp.*)

ARISTOTLE: ON THE HEAVENS. W. K. C. Guthrie. (*2nd Imp.*)

ARISTOTLE: ON THE SOUL, PARVA NATURALIA, ON BREATH. W. S. Hett. (*2nd Imp. revised.*)

ARISTOTLE: ORGANON. H. P. Cooke and H. Tredennick. 3 Vols. Vol. I. (*2nd Imp.*)

ARISTOTLE: PARTS OF ANIMALS. A. L. Peck; MOTION AND PROGRESSION OF ANIMALS. E. S. Forster. (*2nd Imp.*)

ARISTOTLE: PHYSICS. Rev. P. Wicksteed and F. M. Cornford. 2 Vols. (*2nd Imp.*)

ARISTOTLE: POETICS and LONGINUS. W. Hamilton Fyfe; DEMETRIUS ON STYLE. W. Rhys Roberts. (*4th Imp. revised.*)

ARISTOTLE: POLITICS. H. Rackham. (*3rd Imp.*)
ARISTOTLE: PROBLEMS. W. S. Hett. 2 Vols. (Vol. I *2nd Imp. revised.*)
ARISTOTLE: RHETORICA AD ALEXANDRUM. H. Rackham. (With Problems, Vol. II.)
ARRIAN: HISTORY OF ALEXANDER AND INDICA. Rev. E. Iliffe Robson. 2 Vols. (*2nd Imp.*)
ATHENAEUS: DEIPNOSOPHISTAE. C. B. Gulick. 7 Vols. (Vols. I, V and VI *2nd Imp.*)
ST. BASIL: LETTERS. R. J. Deferrari. 4 Vols. (Vols. I, II and IV *2nd Imp.*)
CALLIMACHUS AND LYCOPHRON. A. W. Mair; ARATUS. G. R. Mair.
CLEMENT OF ALEXANDRIA. Rev. G. W. Butterworth. (*2nd Imp.*)
COLLUTHUS. *Cf.* OPPIAN.
DAPHNIS AND CHLOE. *Cf.* LONGUS.
DEMOSTHENES I: OLYNTHIACS, PHILIPPICS AND MINOR ORATIONS: I-XVII AND XX. J. H. Vince.
DEMOSTHENES II: DE CORONA AND DE FALSA LEGATIONE. C. A. Vince and J. H. Vince. (*2nd Imp. revised.*)
DEMOSTHENES III: MEIDIAS, ANDROTION, ARISTOCRATES, TIMOCRATES, ARISTOGEITON. J. H. Vince.
DEMOSTHENES IV-VI: PRIVATE ORATIONS AND IN NEAERAM. A. T. Murray. (Vol. IV *2nd Imp.*)
DEMOSTHENES VII: FUNERAL SPEECH, EROTIC ESSAY, EXORDIA AND LETTERS. N. W. and N. J. DeWitt.
DIO CASSIUS: ROMAN HISTORY. E. Cary. 9 Vols. (Vols. I and II *2nd Imp.*)
DIO CHRYSOSTOM. 5 Vols. Vols I and II. J. W. Cohoon. Vol. III. J. W. Cohoon and H. Lamar Crosby. Vols. IV and V. H. Lamar Crosby. (Vols. I and II *2nd Imp.*)
DIODORUS SICULUS. 12 Vols. Vols. I-V. C. H. Oldfather. Vol. IX. Russel M. Geer. (Vol. I *2nd Imp.*)
DIOGENES LAERTIUS. R. D. Hicks. 2 Vols. (Vol. I *4th Imp.*, Vol. II *3rd Imp.*)
DIONYSIUS OF HALICARNASSUS: ROMAN ANTIQUITIES. Spelman's translation revised by E. Cary. 7 Vols. (Vol. IV *2nd Imp.*)
EPICTETUS. W. A. Oldfather. 2 Vols. (Vol. I *2nd Imp.*)
EURIPIDES. A. S. Way. 4 Vols. (Vol. I *7th Imp.*, Vols. II-IV *6th Imp.*) Verse trans.

EUSEBIUS: ECCLESIASTICAL HISTORY. Kirsopp Lake and J. E. L. Oulton. 2 Vols. (Vol. I *2nd Imp.*, Vol. II *3rd Imp.*)

GALEN: ON THE NATURAL FACULTIES. A. J. Brock. (*3rd Imp.*)

THE GREEK ANTHOLOGY. W. R. Paton. 5 Vols. (Vols. I and II *4th Imp.*, Vols. III and IV *3rd Imp.*)

THE GREEK BUCOLIC POETS (THEOCRITUS, BION, MOSCHUS). J. M. Edmonds. (*7th Imp. revised.*)

GREEK ELEGY AND IAMBUS WITH THE ANACREONTEA. J. M. Edmonds. 2 Vols. (Vol. I *2nd Imp.*)

GREEK MATHEMATICAL WORKS. Ivor Thomas. 2 Vols. (*2nd Imp.*)

HERODES. *Cf.* THEOPHRASTUS: CHARACTERS.

HERODOTUS. A. D. Godley. 4 Vols. (Vols. I-III *4th Imp.*, Vol. IV *3rd Imp.*)

HESIOD AND THE HOMERIC HYMNS. H. G. Evelyn White. (*7th Imp. revised and enlarged.*)

HIPPOCRATES AND THE FRAGMENTS OF HERACLEITUS. W. H. S. Jones and E. T. Withington. 4 Vols. (Vol. I *3rd Imp.*, Vols. II-IV *2nd Imp.*)

HOMER: ILIAD. A. T. Murray. 2 Vols. (*6th Imp.*)

HOMER: ODYSSEY. A. T. Murray. 2 Vols. (*7th Imp.*)

ISAEUS. E. S. Forster. (*2nd Imp.*)

ISOCRATES. George Norlin and LaRue Van Hook. 3 Vols.

ST. JOHN DAMASCENE: BARLAAM AND IOASAPH. Rev. G. R. Woodward and Harold Mattingly. (*2nd Imp. revised.*)

JOSEPHUS. H. St. J. Thackeray and Ralph Marcus. 9 Vols. Vols. I-VII. (Vols. I, V and VI *2nd Imp.*)

JULIAN. Wilmer Cave Wright. 3 Vols. (Vol. I *2nd Imp.*, Vol. II *3rd Imp.*)

LONGUS: DAPHNIS AND CHLOE. Thornley's translation revised by J. M. Edmonds; and PARTHENIUS. S. Gaselee. (*3rd Imp.*)

LUCIAN. A. M. Harmon. 8 Vols. Vols. I-V. (Vols. I and II *2nd Imp.*, Vol. III *3rd Imp.*)

LYCOPHRON. *Cf.* CALLIMACHUS.

LYRA GRAECA. J. M. Edmonds. 3 Vols. (Vol. I *3rd Imp.*, Vol. II *2nd Ed. revised and enlarged*, Vol. III *3rd Imp. revised.*)

LYSIAS. W. R. M. Lamb. (*2nd Imp.*)

MANETHO. W. G. Waddell; PTOLEMY: TETRABIBLOS. F. E. Robbins. (*2nd Imp.*)

MARCUS AURELIUS. C. R. Haines. (*3rd Imp. revised.*)
MENANDER. F. G. Allinson. (*2nd Imp. revised.*)
MINOR ATTIC ORATORS. 2 Vols. Vol. I (Antiphon, Andocides). K. J. Maidment.
NONNOS: DIONYSIACA. W. H. D. Rouse. 3 Vols. (Vol. III *2nd Imp.*)
OPPIAN, COLLUTHUS, TRYPHIODORUS. A. W. Mair.
PAPYRI. NON-LITERARY SELECTIONS. A. S. Hunt and C. C. Edgar. 2 Vols. (Vol. I *2nd Imp.*) LITERARY SELECTIONS. Vol. I (Poetry). D. L. Page. (*3rd Imp.*)
PARTHENIUS. *Cf.* LONGUS.
PAUSANIAS: DESCRIPTION OF GREECE. W. H. S. Jones. 5 Vols. and Companion Vol. arranged by R. E. Wycherley. (Vols. I and III *2nd Imp.*)
PHILO. 11 Vols. Vols. I–V. F. H. Colson and Rev. G. H. Whitaker; Vols. VI–IX. F. H. Colson. (Vols. I, II, V, VI and VII *2nd Imp.*, Vol. IV *3rd Imp. revised.*)
PHILOSTRATUS: THE LIFE OF APOLLONIUS OF TYANA. F. C. Conybeare. 2 Vols. (*3rd Imp.*)
PHILOSTRATUS: IMAGINES; CALLISTRATUS: DESCRIPTIONS. A. Fairbanks.
PHILOSTRATUS AND EUNAPIUS: LIVES OF THE SOPHISTS. Wilmer Cave Wright. (*2nd Imp.*)
PINDAR. Sir J. E. Sandys. (*7th Imp. revised.*)
PLATO: CHARMIDES, ALCIBIADES, HIPPARCHUS, THE LOVERS, THEAGES, MINOS AND EPINOMIS. W. R. M. Lamb.
PLATO: CRATYLUS, PARMENIDES, GREATER HIPPIAS, LESSER HIPPIAS. H. N. Fowler. (*3rd Imp.*)
PLATO: EUTHYPHRO, APOLOGY, CRITO, PHAEDO, PHAEDRUS. H. N. Fowler. (*9th Imp.*)
PLATO: LACHES, PROTAGORAS, MENO, EUTHYDEMUS. W. R. M. Lamb. (*2nd Imp. revised.*)
PLATO: LAWS. Rev. R. G. Bury. 2 Vols. (*2nd Imp.*)
PLATO: LYSIS, SYMPOSIUM, GORGIAS. W. R. M. Lamb. (*4th Imp. revised.*)
PLATO: REPUBLIC. Paul Shorey. 2 Vols. (Vol. I *4th Imp.*, Vol. II *3rd Imp.*)
PLATO: STATESMAN, PHILEBUS. H. N. Fowler; ION. W. R. M. Lamb. (*3rd Imp.*)
PLATO: THEAETETUS AND SOPHIST. H. N. Fowler. (*3rd Imp.*)
PLATO: TIMAEUS, CRITIAS, CLITOPHO, MENEXENUS, EPISTULAE. Rev. R. G. Bury. (*2nd Imp.*)

PLUTARCH: MORALIA. 14 Vols. Vols. I-V. F. C. Babbitt; Vol. VI. W. C. Helmbold; Vol. X. H. N. Fowler. (Vols. I, III and X *2nd Imp.*)
PLUTARCH: THE PARALLEL LIVES. B. Perrin. 11 Vols. (Vols. I, II and VII *3rd Imp.*, Vols. III, IV, VI, VIII-XI *2nd Imp.*)
POLYBIUS. W. R. Paton. 6 Vols.
PROCOPIUS: HISTORY OF THE WARS. H. B. Dewing. 7 Vols. (Vol. I *2nd Imp.*)
PTOLEMY: TETRABIBLOS. *Cf.* MANETHO.
QUINTUS SMYRNAEUS. A. S. Way. (*2nd Imp.*) Verse trans.
SEXTUS EMPIRICUS. Rev. R. G. Bury. 4 Vols. (Vols. I and III *2nd Imp.*)
SOPHOCLES. F. Storr. 2 Vols. (Vol. I *8th Imp.*, Vol. II *5th Imp.*) Verse trans.
STRABO: GEOGRAPHY. Horace L. Jones. 8 Vols. (Vols. I and VIII *3rd Imp.*, Vols. II, V and VI *2nd Imp.*)
THEOPHRASTUS: CHARACTERS. J. M. Edmonds; HERODES, etc. A. D. Knox. (*2nd Imp.*)
THEOPHRASTUS: ENQUIRY INTO PLANTS. Sir Arthur Hort. 2 Vols. (*2nd Imp.*)
THUCYDIDES. C. F. Smith. 4 Vols. (Vol. I *3rd Imp.*, Vols. II-IV *2nd Imp. revised.*)
TRYPHIODORUS. *Cf.* OPPIAN.
XENOPHON: CYROPAEDIA. Walter Miller. 2 Vols. (Vol. I *2nd Imp.*, Vol. II *3rd Imp.*)
XENOPHON: HELLENICA, ANABASIS, APOLOGY, AND SYMPOSIUM. C. L. Brownson and O. J. Todd. 3 Vols. (Vols. I and III *3rd Imp.*, Vol. II *4th Imp.*)
XENOPHON: MEMORABILIA AND OECONOMICUS. E. C. Marchant. (*2nd Imp.*)
XENOPHON: SCRIPTA MINORA. E. C. Marchant. (*2nd Imp.*)

VOLUMES IN PREPARATION

GREEK AUTHORS

ARISTOTLE: DE MUNDO, etc. D. Furley and E. S. Forster.
ARISTOTLE: HISTORY OF ANIMALS. A. L. Peck.

ARISTOTLE: METEOROLOGICA. H. D. P. Lee.
PLOTINUS.

LATIN AUTHORS

ST. AUGUSTINE: CITY OF GOD.
[CICERO:] AD HERENNIUM. H. Caplan.
CICERO: PRO SESTIO, IN VATINIUM, PRO CAELIO, DE PROVINCIIS CONSULARIBUS, PRO BALBO. J. H. Freese and R. Gardner.
PHAEDRUS AND OTHER FABULISTS. B. E. Perry.

DESCRIPTIVE PROSPECTUS ON APPLICATION

LONDON
WILLIAM HEINEMANN LTD
Cloth 15s.

CAMBRIDGE, MASS.
HARVARD UNIV. PRESS
Cloth $2.50

CENTRAL ARCHAEOLOGICAL LIBRARY,
NEW DELHI
Issue Record
No. 510.949/the